A Sufficient Statistic for Influence in Structured Multiagent Environments

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Abstract

Making decisions in complex environments is a key challenge in artificial intelligence (AI). Situations involving multiple decision makers are particularly complex, leading to computation intractability of principled solution methods. A body of work in AI [4, 3, 41, 45, 47, 2] has tried to mitigate this problem by trying to bring down interaction to its core: how does the policy of one agent *influence* another agent? If we can find more compact representations of such influence, this can help us deal with the complexity, for instance by searching the space of influences rather than that of policies [45]. However, so far these notions of influence have been restricted in their applicability to special cases of interaction. In this paper we formalize *influence-based abstraction (IBA)*, which facilitates the elimination of latent state factors *without any loss in value*, for a very general class of problems described as factored partially observable stochastic games (fPOSGs) [33]. This generalizes existing descriptions of influence, and thus can serve as the foundation for improvements in scalability and other insights in decision making in complex settings.

Contents

1. Introduction

One of the important ideas in the development of algorithms for multiagent systems (MASs) is the identification of compressed representations of the information that is relevant for an agent [4, 3, 41, 36, 44, 45, 46, 42, 48, 47, 33]. For instance, when a cook and a waiter collaborate, the waiter might not need to know all details of how the cook prepares the food; it may be sufficient if it has an understanding of the time that it will take.

In this paper we further investigate so-called *influence-based abstractions,* which aim at decomposing structured MASs into a set of smaller interacting problems [33]. In particular, we describe the concept of *influence-based abstraction (IBA)*, which facilitates the abstraction of latent state variables without sacrificing reward (task performance), in detail. It constructs a smaller, local model for one of the agents given the policies of the other agents, which can subsequently be used to compute a best response. IBA consists of two steps: first, we compute a so-called *influence point*—a more abstract representation of how the agents local problem is affected by other agents and external (i.e., non-local) parts of the problem—, second, this influence is used to construct the smaller *influence-augmented local model (IALM).*

IBA does not only give a new perspective on best-response computations themselves, but also forms the basis of *influence search* [4, 45, 47, 2], which can provide significant speedup for multiagent planning by searching the space of *joint influences*, which can be much smaller than the space of joint policies. It also can serve as the basis of providing guarantees on the quality of heuristic solutions, by considering *optimistic* influences [31], and form an inspiration for neural network architectures in deep reinforcement learning, that compute approximate versions of influence that can improve learning, both in terms of speed as performance prove a stronger [10].

This document gives a formal definition of influence that can be used to perform IBA for general factored partially observable stochastic games (fPOSGs) [14, 7], and proves that an IALM constructed using this definition of influence in fact allows to compute an *exact* best-response. In other words, that this description of influence is a *sufficient statistic* of the policy of the other agents in order to predict the optimal value. It extends our previous paper [33] in the following ways:

1. it provides a complete proof of the claimed exactness of IBA;
2. it elaborates on a number of technical subtleties, such as dealing with multiple sources of influence, and specifying initial beliefs in the IALM;
3. it provides an extension of IBA and corresponding proofs to fPOSGs with intra-stage dependencies, which are critical for the expressiveness of the formalism (cf. Section 4.1.1);
4. it provides additional illustration and explanation, making the concept of IBA more accessible.

Additionally, we make a simple (but novel in the context of IBA) observation: the presence of other agents can be seen as a generalization of single agent settings, which directly implies that *our formulation of IBA also provides a sufficient statistic for decision making under abstraction for a single agent*. While there is a multitude of performance loss bounds available for abstractions, e.g. see [12, 11, 13, 15, 20, 35, 1], these are usually based on assumed quality bounds on the transition probabilities and rewards of the abstracted model. In contrast, our work here shows how an abstracted model can preserve exact transition and reward predictions, by 'remembering' appropriate elements of the local history. In the words of McCallum [21], we detail an approach to *perfectly* "uncover [...] hidden state" in abstractions for a large class of structured problems.

As such, the contributions of this paper are of a theoretical nature: it provides a principled understanding of lossless abstractions in structured (multiagent) decision problems, it formalizes this, and extends the scope of applicability. The main technical result is the proof sufficiency of IBA given in Section 6. The proof is not only a validation of the theory, it also serves a practical purpose: it isolates the core technical property that needs to hold for sufficiency, thus providing 1) insight into *how* abstraction of state latent factors affects value, 2) a derivation that can be used to obtain a simplification of influence in simpler cases, and 3) a recipe of how to prove similar results more complex settings.

This paper is organized as follows: First, Section 2 provides the necessary background by introducing single and multiagent models for decision making. Section 3 introduces the concept of computing best responses (using global value functions) to the policies of other agents and the concept of 'local form models' which formalizes a desired abstraction for an agent. Next, in Section 4, we bring these concepts together: we show how an agent can locally compute a best-response (compute a local value function) provided it is given an influence point. Section 5 extends this description to problems with intra-stage dependencies. Section 6 then presents the main proof of sufficiency of our influence points, i.e., it shows that they are sufficient to predict the best-response value without any loss in value. Finally, Section 7 concludes.

2. Background

Here we concisely provide background on some of the (both single- and multi-agent) models that we use. The main purpose is to introduce the notation formally. For an extensive introduction to *partially observable Markov decision processes (POMDPs)* we refer to [17, 40], for an introduction to multiagent variants see [39, 26, 28].

2.1 Single-Agent Models: POMDPs

Partially observable Markov decision processes, or POMDPs, provide a formal framework for the interaction of an agent with a stochastic, partially observable environment. That is, it provides an agent with the capabilities to reason about both action uncertainty, as well as state uncertainty.

2.1.1 Model

A POMDP is a discrete time model, in which the agent selects an action at every time step or *stage.* It extends the regular *Markov decision process (MDP)* [38] to settings in which the state of the environment cannot be observed. It can be formally defined as follows.

Definition 1 (POMDP)**.** A *partially observable Markov decision process (POMDP)* is defined as a tuple $\mathcal{M}^{POMDP} = \langle \mathcal{S},\mathcal{A},T,R,\mathcal{O},O,h,b^0 \rangle$ with the following components:

- $\mathcal{S}$ is a (finite) set of states;
- $\mathcal{A}$ is the (finite) set of actions;
- T is the transition probability function, that specifies $T(s,a,s') = \Pr(s'|s,a)$, the probability of a next state s' given a current state s and action a;
- R is the immediate reward function $R : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$. With $R(s,a,s')$ we denote the reward specified for a particular transition s,a,s';
- $\mathcal{O}$ is the set of observations;
- O is the observation probability function, which specifies $O(a^t,s^{t+1},o^{t+1}) = \Pr(o|a,s')$, the probability of a particular observation o after a and resulting state s';
- h is the horizon of the problem as mentioned above;
- $b^0 \in \Delta(\mathcal{S})$, is the initial state distribution at time $t = 0$.[1]

In many cases, the set of states is huge, and states can be thought of as composed of values assigned to different variables:

Definition 2 (Factored POMDP)**.** When the state space is spanned by a set of state variables, or *factors*, we call the problem a *factored POMDP (fPOMDP).*

The merit of such a factored POMDP is that, by making the structure of the problem (i.e., how different factors influence each other) explicit, the model can be much more compact. In particular, the initial state distribution can be compactly represented as a *Bayesian network* [34, 6, 19], and the transition and reward model can be specified compactly using a *two-stage dynamic Bayesian network (2DBN)* [7], and a similar approach can be taken for the observation model [37]. (An example of a 2DBN will be discussed in Fig. 2 on page 7.)

2.1.2 Beliefs

In contrast to regular MDPs, in a POMDP the agent cannot observe the state; it only observes the observations. However, the observations are not a Markovian signal: i.e., the last observation o^t made by the agent does not provide the same amount of information (to predict the rewards and the future of the the process) as the *action-observation history (AOH)*, the entire history of actions and observations $\vec{\theta}^t = (a^0,o^1,\ldots,a^{t-1},o^t)$. This means that the agent needs to select its actions based on the history of observations.

Luckily, for a POMDP this history can be summarized compactly as a *belief,* which is defined as the posterior probability distribution over states given the history:

$$b(s) \triangleq \Pr(s|b^0,\vec{\theta}^t).$$

1. $\Delta(\cdot)$ denotes the set of probability distributions over $(\cdot)$.

The belief does not only summarize the history, it does so in a lossless way. That is, a belief is a *sufficient statistic* for optimal decision making [5]; it allows an agent to reach the same performance as an agent that would act optimally based on the AOH $\vec{\theta}^t$.

This belief can be recursively computed, which means that an agent can update its belief as it interacts with its environment. We write $b' = BU(b,a,o)$, where $BU(b,a,o)$ is the belief update operator that, given a previous belief b taken action a and received observation o, produces the next belief:

$$BU(b,a,o)(s') = \frac{1}{\Pr(o|b,a)} \Pr(o|a,s') \sum_s \Pr(s'|s,a)b(s). \tag{2.1}$$

Here, $\Pr(o|b,a)$ is a normalization constant:

$$\Pr(o|b,a) = \mathbf{E}_{s\sim b,s'\sim T(s,a,\cdot)} [O(a,s',o)] = \sum_{s'} \Pr(o|a,s') \sum_s \Pr(s'|s,a)b(s).$$

2.1.3 Policies & Value Functions

In a POMDP, the agent employs a *policy,* π, to interact with its environment. Such a policy is a (deterministic) mapping from beliefs to actions. Note that, given the initial belief b^0, such a policy will specify an action for each observation history.[2]

The goal of the decision maker, or agent, in the POMDP is to maximize the expected (discounted) cumulative reward:

$$\mathbf{E}\left[\sum_{t=0}^{h-1} \gamma^t R(s^t,a^t,s^{t+1})|b^0,\pi\right],$$

here

- h is the horizon, i.e., the number of time steps, or *stages,* for which we want to plan,
- the expectation is over sequences of states and observations induced by the policy π,
- $\gamma \in [0,1]$ is the discount factor.

In this work, we focus on the finite-horizon case, in which it is typical (but not necessary) to assume $\gamma = 1$.

For a finite-horizon POMDP, the optimal value function for stage t can be expressed as

$$Q^t(b,a) = R(b,a) + \gamma \sum_o \Pr(o|b,a)V^{t+1}(BU(b,a,o))$$

where $V^{t+1}(b') = \max_{a'} Q^{t+1}(b',a')$ is the value of acting optimally in the next time step and $R(b,a)$ is the expected immediate reward:

$$R(b,a) = \mathbf{E}_{s^t\sim b,s^{t+1}\sim T(s^t,a^t,\cdot)} \left[R(s^t,a^t,s^{t+1})\right] = \sum_s b(s) \sum_{s'} \Pr(s'|s,a)R(s,a,s').$$

2.2 Multiagent Models: POSGs

The POMDP model can be extended to include multiple self-interested agents as follows.

Definition 3 (POSG)**.** A *partially observable stochastic game (POSG)* is defined as a tuple $\mathcal{M}^{POSG} = \left\langle \mathcal{D},\mathcal{S},\mathcal{A},T,\mathcal{R},\mathcal{O},O,h,b^0 \right\rangle$ with the following components:

- $\mathcal{D} = \{1,\ldots,n\}$ is the set of n agents.
- $\mathcal{S}$ is a (finite) set of states.

2. This can be seen as follows: for b^0 the policy specifies an action, a^0, then given o^1 we can compute b^1 which we can use to look up a^1, etc.

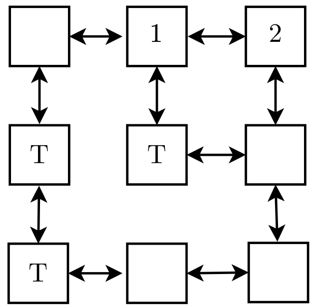


Figure 1: A possible instantiation of the HOUSE SEARCH problem: 1, 2 represent the starting locations of the agents, while 'T' encodes the possible locations of the target.

- $\mathcal{A} = \mathcal{A}_1 \times \ldots \mathcal{A}_n$ is the set of *joint* actions $a = \langle a_1, \ldots, a_n \rangle$, with $\mathcal{A}_i$ the set of individual actions for agent i.
- T is the transition probability function.
- $\mathcal{R} = \langle R_1, \ldots R_n \rangle$ is the collection of immediate reward function for each agent. Each $R_i : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$ maps from states, joint actions and next states to an immediate reward for that agent.
- $\mathcal{O} = \mathcal{O}_1 \times \cdots \times \mathcal{O}_n$ is the set of joint observations, with $\mathcal{O}_i$ the set of individual observations for agent i,
- O is the observation probability function, which specifies $\Pr(o|a,s')$, the probability of a particular joint observation o after a and resulting state s'.
- h is the horizon of the problem as mentioned above.
- $b^0 \in \Delta(\mathcal{S})$, is the initial state distribution at time $t = 0$.

Since in a POSG each agent has its own goal, there no longer is a definition of optimality. Instead it is customary to focus on game-theoretic solution concepts for POSGs [14]. Such solutions, e.g., Nash equilibria, typically specify a tuple of policies $\pi = \langle \pi_1, \ldots, \pi_n \rangle$, one for each agent, that are in equilibrium. In general, we will refer to a tuple of policies π as a *joint policy.*

Of course, it is also possible to consider cooperative teams of agents. In this case, we align the goals of the agents by giving them the same reward function:

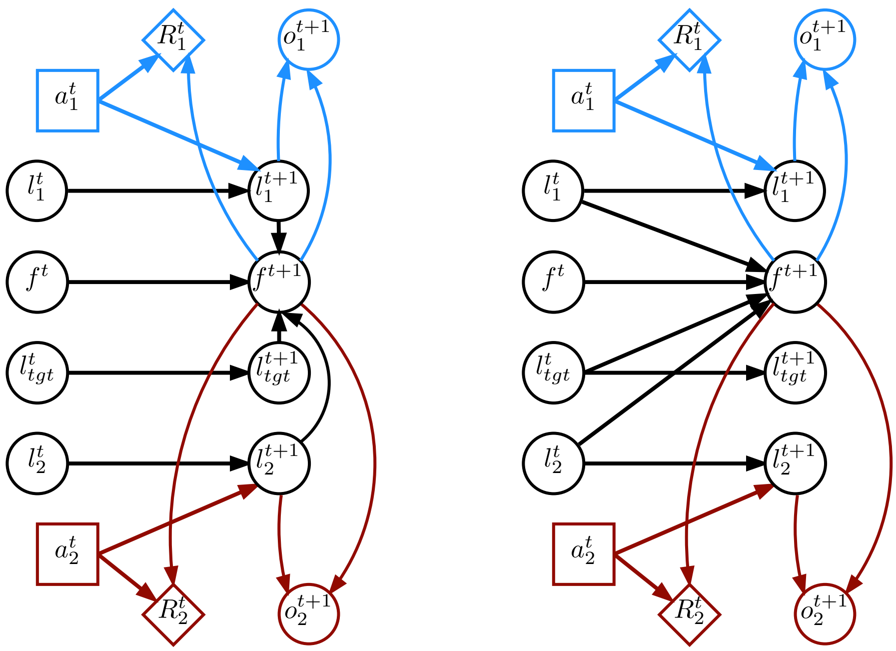
Definition 4 (Dec-POMDP)**.** A *decentralized partially observable Markov decision process (Dec-POMDP)* is a POSG where all agents share the same reward function: $\forall_{i,j}\ R_i = R_j$.

Since interests are aligned, in a Dec-POMDP we can speak about optimality. Moreover, there is guaranteed to be at least one *deterministic* joint policy that is optimal [29]. As was the case for POMDPs, we can also consider variants of the multiagent models with factored state spaces. We will refer to these as *factored POSGs (fPOSGs)* and *factored Dec-POMDPs (fDec-POMDPs)* [30].[3]

As an example, we consider the HOUSE SEARCH problem [32], in which a team of robots must find a target (say a remote control) in a house with multiple rooms. This task is representative of an important class of problems in which a team of agents needs to locate objects or targets. In HOUSE SEARCH the assumption is that a prior probability distribution over the location of the target is available and that the target is stationary or moves in a manner that does not depend on the strategy used by the searching agents.

3. More recently, researchers have also investigated deterministic and non-deteterministic versions, called (factored) qualitative Dec-POMDP [9]. We will not particularly target this special case in this paper, but note that ideas of influence search can be exploited in this context too [2].

Example. The HOUSE SEARCH environment can be represented by a graph, as illustrated in Fig. 1. At every time-step each agent can stay in the current room or move to a next one. The location of an agent i at time step t is denoted l_i^t and that of the target is denoted l_{target}^t. The actions (movements) of each agent have a specific cost $c_i(l_i,a_i)$ (e.g., the energy consumed by navigating to a next room) and can fail; we allow for stochastic transitions $p(l_i'|l_i,a_i)$. Also, each robot might receive a penalty c_{time} for every time step that the target is not found yet. When a robot is in (or near) the same node as the target, there is a probability of detecting the target $p(detect_i|l_{target},l_i)$, which will be modeled by a boolean state variable 'target found' f^t, which both agents can observe (thus modeling a communication channel which the agents can only use to inform each other of detection).When the target is detected, the agents also receive a reward r_{detect}. Given the prior distribution and model of target behavior, the goal is to optimize the sum (over time) of rewards, thus trading off movement cost and probability of detecting the target as soon as possible. We will focus on the situation where each agent has its individual rewards (so the POSG setting). In previous work, the house search problem was treated as a Dec-POMDP by defining the team reward as the sum of the individual rewards [32].



(a) With intra-stage connections. (b) Without intra-stage connections.

Figure 2: Factored representation of the HOUSE SEARCH problem. Actions, observations and rewards of the first agent are in light blue, while those of agent 2 are in dark red. State variables are in black.

Fig. 2a demonstrates how a 2DBN can be used to compactly represent the transition, observation and reward model. For instance, for each state variable at a state $t+1$, the 2DBN shows which other entities (state factors and actions) influence it. The figure illustrates that most dependencies are *across-stage* (e.g., l_2^t influences l_2^{t+1}) but that it is also possible to have *intra-stage dependencies (ISDs)*. For instance, whether the target will be detected at stage $t+1$ depends on l_2^{t+1} not on l_2^t. The representation of the transition model is compact since it can be represented as a product of *conditional probability tables (CPTs)*, each of which are exponential only in the number of incoming dependencies. So as long as the number of incoming connections is limited, the transition

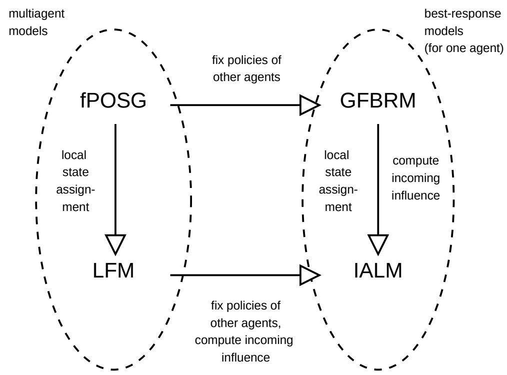


Figure 3: Overview of various models used.

probabilities can be represented compactly. Fig. 2a also shows that this type of representation can also be employed for observation probabilities, as well as rewards.

Since ISDs complicate notation and the definitions of influence, we also consider a version of the problem that has no intra-stage connections, shown in Fig. 2b. For rewards and observations, intra-stage connections are still allowed. (In fact, since the observation probabilities in the standard POMDP definition depend on the next state s', there is no way of representing them without intra-stage connections). Note that this is a slightly different problem than the problem represented in Fig. 2a: in the problem without ISDs the agents have a chance of detecting the target at stage $t+1$ if they are co-located with the target at stage t, which means that there is a one-step delay incurred before they receive the reward. This illustrates the fact the ISDs do allow for a more expressive model, and that therefore developing theory that support such connections is an important goal.

To facilitate easier exposition, in Section 4 we will first introduce the concept of influence-based abstraction without ISDs. These will be considered in Section 5. Before we can jump to the topic of influence-based abstraction, however, we will need to discuss decision problems from a local perspective, in Section 3, which covers problems with ISDs.

3. Best Responses & Local-Form Models

In contrast to the typical solutions to POSGs and Dec-POMDPs, which try and identify a joint policy as the solution, this paper focuses on computing best-response policies in interactive settings. That is, given a multiagent model with state uncertainty (either a POSG or Dec-POMDP) and given some policy for the other agents $\pi_{\neq i} = \langle \pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_n \rangle$, we want to compute the best response π_i^{BR} for agent i. Such best-response computation is obviously important for self-interested agents (i.e., in POSGs), but is also an important component in many Dec-POMDP solution methods [24, 25, 18].

As illustrated in Fig. 3, we will consider a number of different types of models in this paper. The starting point is given by the fPOSG or a special case thereof (e.g., a Dec-POMDP). We refer those models as global-form models (GFMs). For such models, it is possible to directly compute a best-response by fixing the policies of the other agents. We refer to the resulting POMDP as a global-form best-response model (GFBRM); these models will be introduced next. Subsequently, we will introduce *local-form models (LFMs)* these models restricts the state factors that each agent primarily cares about. That is, an agent in an LFM only reasons about a subset of factors. This will then form the basis for computing best-responses in such a local model, called *influence-augmented local model (IALM),* which will be enabled by influence-based abstraction introduced in Section 4.

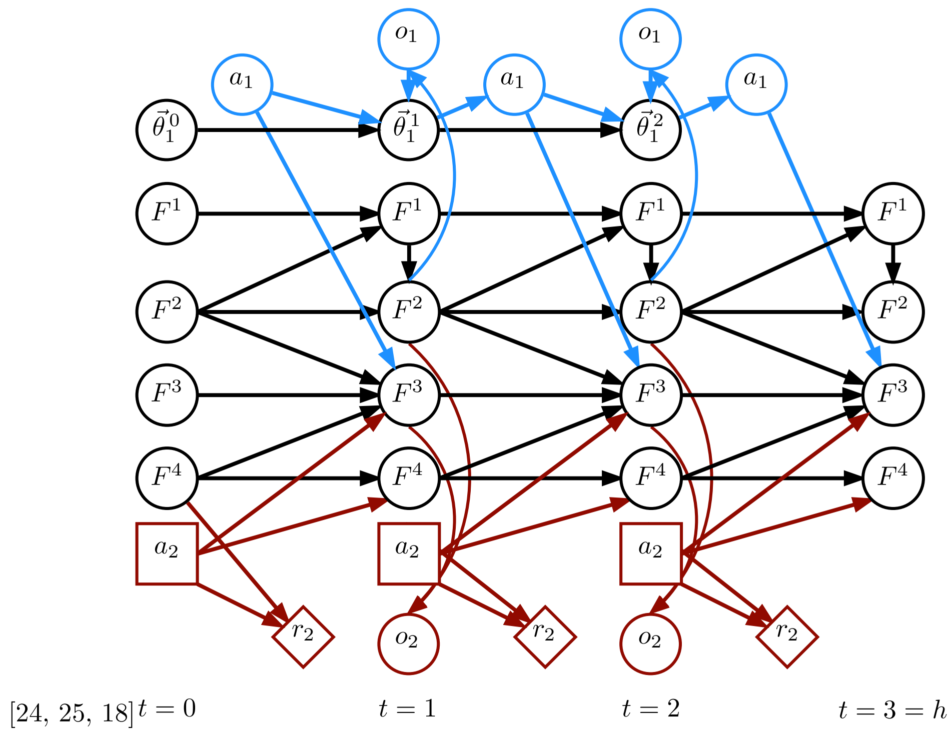


Figure 4: A hypothetical global-form best-response model for agent 2, unrolled over time. The action-observation history (i.e., internal state) of agent 1 can be interpreted as a state factor in this model.

3.1 Global-Form Best-Response Model

In this section we define a Global-Form Best-Response Model (GFBRM) that an agent can use in order to compute a best-response in a general POSG. We first define this model and then talk about value functions for this model.[4]

Model The basic idea of defining a best-response model is shown in Fig. 4. By fixing $\pi_{\neq i}$, the policies of the other agents, all the choice nodes are turned into random variables that now depend on the AOHs that those agents observed. So the key construct here is that the AOH of the other agent(s) is made part of the state of the best-response model. This can be formalized as follows.

Definition 5 (Global-Form Best-Response Model)**.** Let $\mathcal{M}^{POSG} = \left\langle \mathcal{D},\mathcal{S},\mathcal{A},T,\mathcal{R},\mathcal{O},O,h,b^0 \right\rangle$ be a (f)POSG and let $\pi_{\neq i}$ be a profile of policies for all agents but i. A *Global-Form Best-Response Model (GFBRM)* for agent i is a POMDP: $\mathcal{M}_i^{GFBR}(\mathcal{M}^{POSG},\pi_{\neq i}) = \left\langle \bar{\mathcal{S}},\mathcal{A}_i,\bar{T}_i,\bar{R}_i,\mathcal{O}_i,\bar{O}_i,h,\bar{b}_i^{\,0} \right\rangle$, where

- $\bar{\mathcal{S}}$ is the set of augmented states $\bar{s}_i^t = \langle s,\vec{\theta}_{\neq i}^{\,t} \rangle$ that specify a underlying state of the POSG as well as an AOH history for all the other agents.
- $\mathcal{A}_i,\mathcal{O}_i$ are the (unmodified) sets of actions and observations for agent i.

4. Our formulation here is closely related to the way best-responses are computed in DP-JESP [24]: essentially our representation here is a reformulation that makes explicit the fact that fixing the policies of other agents leads to a single-agent POMDP model.

- The transitions

$$\begin{aligned}\bar{T}(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t) &= \bar{T}(\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1}\rangle|\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i^t) \\ &= \bar{T}(\langle s^{t+1},\left(\vec{\theta}_{\neq i}^t,a_{\neq i}^t,o_{\neq i}^{t+1}\right)\rangle|\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i^t) \\ &= \Pr(o_{\neq i}^{t+1},s^{t+1},a_{\neq i}^t|s^t,a_i^t,\vec{\theta}_{\neq i}^t) \\ &= \Pr(o_{\neq i}^{t+1}|a_i^t,a_{\neq i}^t,s^{t+1})\Pr(s^{t+1}|s^t,a_i^t,a_{\neq i}^t)\Pr(a_{\neq i}^t|\vec{\theta}_{\neq i}^t) \\ &= \left[\sum_{o_i^{t+1}} O(o^{t+1}|a^t,s^{t+1})\right] T(s^{t+1}|s^t,a^t)\pi_{\neq i}(a_{\neq i}^t|\vec{\theta}_{\neq i}^t)\end{aligned}$$

  with $\pi_{\neq i}(a_{\neq i}^t|\vec{\theta}_{\neq i}^t) = \prod_{j\neq i}\pi_j(a_j^t|\vec{\theta}_j^t)$ the probability of $a_{\neq i}^t$ given $\vec{\theta}_{\neq i}^t$ according to $\pi_{\neq i}$.
- The observations

$$\begin{aligned}\bar{O}(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1}) &= \bar{O}(o_i^{t+1}|a_i^t,\langle s^{t+1},\left(\vec{\theta}_{\neq i}^t,a_{\neq i}^t,o_{\neq i}^{t+1}\right)\rangle) \\ &= \Pr(o_i^{t+1}|a_i^t,a_{\neq i}^t,s^{t+1},o_{\neq i}^{t+1}) \\ &= \frac{\Pr(o_i^{t+1},o_{\neq i}^{t+1}|a_i^t,a_{\neq i}^t,s^{t+1})}{\Pr(o_{\neq i}^{t+1}|a_i^t,a_{\neq i}^t,s^{t+1})} \\ &= \frac{O(o^{t+1}|a^t,s^{t+1})}{\sum_{o_i^{t+1}} O(o^{t+1}|a^t,s^{t+1})}\end{aligned}$$

- $\bar{R}_i$ is the augmented reward model

$$\begin{aligned}\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1}) &= \bar{R}_i(\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i^t,\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1} = \left(\vec{\theta}_{\neq i}^t,a_{\neq i}^t,o_{\neq i}^{t+1}\right)\rangle) \\ &= R_i(s^t,a_i^t,a_{\neq i}^t,s^{t+1})\end{aligned}$$

  Note that $a_{\neq i}^t$ is specified by $\bar{s}_i^{t+1}$.
- h is the (unmodified) horizon.
- $\bar{b_i}^0$ is the initial belief.

A GFBRM is a POMDP, which means that an agent can track a belief, which is now a distribution over *augmented states* $\bar{s}_i = \langle s^t,\vec{\theta}_{\neq i}^t\rangle$, as usual. We will refer to such beliefs as *global-form beliefs,* denoted b_i^g. The initial global-form belief follows directly from the initial belief of the POSG. Since at the first stage, the history of the other agents is the empty history, it is trivially constructed from b^0: $\forall_s\ b_i^g(\langle s,()\rangle) = b^0(s)$.

Note that the description of the GFBRM depends rather crucially on the fact that we choose AOHs for the representation of the internal state of the other agent(s). That is, we assume that the policies of the other agent(s) are based on their AOHs. While this is a very general model, other models of other agents with a more limited description of internal state can be useful too. For such more compact descriptions, however, it is not always possible to construct a POMDP model with an independent transition and observation model. Instead, one may need to replace $\bar{T}$,$\bar{O}$ by a combined 'dynamics function' $\bar{D}$ that specifies $\bar{D}(\bar{s}_i^{t+1},o_i^{t+1}|\bar{s}_i^t,a_i^t)$. For more details see [27].[5]

5. Essentially in such a setting we have that augmented states are tuples of nominal states and internal states of other agents $\bar{s}_i^t = \langle s^t,I_{\neq i}^t\rangle$. The internal states of the other agent are updated based upon the taken actions and observations, but do not store those actions and observations. This means that, in general, $\bar{D}$ is specified as a

Value Function Since a GFBRM is just a POMDP, all POMDP theory and solution methods apply. E.g., the value function is given by:

$$Q_i(b_i^g,a_i) = R_i(b_i^g,a_i) + \gamma \sum_{o_i} \Pr(o_i|b_i^g,a_i) V_i(BU(b_i^g,a_i,o_i)) \tag{3.1}$$

where

$$\begin{aligned} R_i(b_i^g,a_i^t) &= \mathbf{E}_{\bar{s}_i^t \sim b_i^g, \bar{s}_i^{t+1} \sim \bar{T}_i(\bar{s}_i^t,a_i^t,\cdot)} \left[\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1})\right] \\ &= \sum_{s^t} \sum_{s^{t+1}} \sum_{a_{\neq i}} \Pr(s^{t+1}|s^t,a) R_i(s^t,a,s^{t+1}) \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t) b_i^g(s^t,\vec{\theta}_{\neq i}^t) \end{aligned} \tag{3.2}$$

(see Appendix A.1.1) and

$$\begin{aligned} \Pr(o_i^{t+1}|b_i^g,a_i^t) &= \mathbf{E}_{\bar{s}_i^t \sim b_i^g, \bar{s}_i^{t+1} \sim \bar{T}_i(\bar{s}_i^t,a_i^t,\cdot)} \left[\bar{O}_i(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1})\right] \\ &= \sum_{s^t} \sum_{s^{t+1}} \sum_{a_{\neq i}} \sum_{o_{\neq i}^{t+1}} \Pr(s^{t+1}|s^t,a) \Pr(o^{t+1}|a,s^{t+1}) \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t,\pi_{\neq i}) b_i^g(s^t,\vec{\theta}_{\neq i}^t) \end{aligned} \tag{3.3}$$

(see Appendix A.1.2.)

Solution of the GFBRM gives the best-response value for agent i:

$$V_i(\pi_{\neq i}) \triangleq V_i(b_i^{g,0}). \tag{3.4}$$

3.2 Local-Form Model

GFBRMs allow an agent i to compute a best-response policy against the fixed policies $\pi_{\neq i}$ of the other agents. A difficulty here is that agent i needs to reason about many state factors as well as the internal state (the action-observation history) of the other agents. That is, drawing an analogy to human interactions, it is like in a simple collaborative task (e.g., carrying a table), we would need to reason over the inner working of our collaborator's brain, as well as over the sequence of images that he or she perceives. Clearly, such an approach is infeasible in general. To make a step in the direction to overcome this problem, here we introduce *local-form models (LFMs)* which restrict the set of state factors that each agent primarily cares about, and eliminates the dependence on the AOH of other agents.

Local States An LFM augments an fPOSG with a function that provides a description of each agent's *local state*. Local state descriptions comprise potentially overlapping subsets of state factors that will allow us to decompose an agent's best-response computation from the global state. We start with some definitions.

Definition 6 (Local state function)**.** The *local state function* $S : \mathcal{D} \rightarrow 2^{\mathcal{SF}}$ maps from agents to subsets of state factors $S(i)$ that are in the agents local state.

Definition 7 (Observation-relevant factor)**.** We say that a state factor F is *observation-relevant* for an agent i, denoted $\mathrm{ORel}_i(F)$, if it influences the probability of the agent's observation. That is, when in the 2DBN there is a link from F^t to o_i^t (i.e., F is a parent of o_i^t).

marginal:

$$\begin{aligned} \bar{D}(\bar{s}_i^{t+1},o_i^{t+1}|\bar{s}_i^t,a_i^t) &= \Pr(\langle s^{t+1},I_{\neq i}^{t+1}\rangle,o_i^{t+1}|\langle s^t,I_{\neq i}^t\rangle,a_i^t) \\ &= \sum_{a_{\neq i}^t,o_{\neq i}^{t+1}} \Pr(I_{\neq i}^{t+1}|I_{\neq i}^t,a_{\neq i}^t,o_{\neq i}^{t+1}) O(o^{t+1}|a^t,s^{t+1}) T(s^{t+1}|s^t,a^t) \pi_{\neq i}(a_{\neq i}^t|I_{\neq i}^t) \end{aligned}$$

and it is not possible to decompose it into a transition and observation function.

Definition 8 (Reward-relevant Factor)**.** Similarly, a state factor F is *reward-relevant* for an agent i, $\text{RRel}_i(F)$ if it influences the agent's rewards, i.e., if F^t or F^{t+1} is a parent of R_i^t.

We say that a state factor F is *modeled* by an agent i if it is part of its local state space: $F \in S(i)$. We can now define the local-form model.

Definition 9 (Local-form model)**.** A *local-form POSG,* also referred to as *local-form model (LFM),* is a pair $\mathcal{M}^{LFM} = \langle \mathcal{M}, S \rangle$, where $\mathcal{M}$ is an fPOSG and S is a local state function such that, for all agents:

1. All observation-relevant factors are in the local state: $\forall_i \forall_F \ \text{ORel}_i(F) \implies F \in S(i)$.
2. All reward-relevant factors are in the local state: $\forall_i \forall_F \ \text{RRel}_i(F) \implies F \in S(i)$.

The basic idea behind this definition is that we will abstract away all the non-modeled factors for the best-response computation. The requirements on observation- and reward-relevant factors make certain that the observation probabilities and rewards are still specified in this abstracted model. Note also that this means that we will only be able to abstract away (latent) state variables, not observation variables themselves. We will show that such latent factor abstraction can, in principle, be performed without loss in value. This certainly would not be the case for abstracting away observation variables: in general this would lead to a loss of information and a corresponding drop in achievable value.

The focus in this text is on computing a best-response for one agent i. This allows us to divide the set of state factors in ones modeled by agent i's local problem (indicated with x) and ones that are not modeled (indicated with y).[6] In particular, we will write

- x^l (an instantiation of) a modeled factor (with index l),
- x_i (an instantiation of) all modeled factors of agent i,
- y^l (an instantiation of) a non-modeled factor (with index l),
- y_i (an instantiation of) all non-modeled factors of agent i.

Transition probabilities In an LFM, the probability of the next local state is the marginal of the entire state:

$$\Pr(x_i^{t+1} | s^t, a_i, a_{\neq i}) = \sum_{y_i^{t+1}} \Pr(x_i^{t+1}, y_i^{t+1} | s^t, a_i, a_{\neq i}) \tag{3.5}$$

In an LFM, just as in a normal fPOSG, the flat transition probabilities on the right hand side of this equation are given by the product of the CPTs. However, from the perspective of an agent i we can now group these CPTs in three different categories: 1) those corresponding to modeled factors that are only affected by other factors and actions that are modeled, 2) those corresponding to modeled factors that are affected by at least one factor or action of the external problem, and 3) those corresponding to non-modeled factors. We will refer to the state factors corresponding to these as:

1. *Only-locally-affected factors (OLAFs)* xl^k. These can have incoming arrows from all modeled factors x_i^t at the previous stage, and from all modeled factors x_i^{t+1} intra-stage (but, obviously, excluding $xl^{k,t+1}$ itself, and respecting a non-cyclic structure as any 2DBN).
2. *Non-locally-affected factors (NLAFs)* xn^k; These can (also) be affected by any other factor or action, intra-stage or previous stage.
3. *Non-modeled factors (NMFs)* y^k.

6. More generally, from the perspective of agent i, S partitions the modeled factors $S(i)$ in two sets: a set of *private* factors that it models but other agents do not, and a set of *mutually-modeled factors* (MMFs) that are modeled by agent i as well as some other agent j. This distinction plays a crucial role in influence search for TD-POMDPs [44], but is less important for computing best-responses as considered in this document.

These three types of factors are illustrated in Fig. 5a, which shows a hypothetical local-form model. Using the introduced notation, we can write the transition probabilities as:

$$\begin{aligned}\Pr(s^{t+1}|s^t,a_i,a_{\neq i}) &= \left[\Pr(xl_i^{t+1}|\dots)\Pr(xn_i^{t+1}|\dots)\Pr(y_i^{t+1}|\dots)\right]\\ &= \Pr(xl_i^{t+1}|x_i^t,xn_i^{t+1},a_i)\Pr(xn_i^{t+1}|x_i^t,xl_i^{t+1},y_i^t,y_i^{t+1},a_i,a_{\neq i})\Pr(y_i^{t+1}|x_i^t,x_i^{t+1},y_i^t,a_i,a_{\neq i})\end{aligned} \tag{3.6}$$

with

- $\Pr(xl_i^{t+1}|x_i^t,xn_i^{t+1},a_i)$ representing a product of CPTs of OLAFs xl^k:

$$\Pr(xl_i^{t+1}|x_i^t,xn_i^{t+1},a_i) = \prod_{k\in OLAF(i)} \Pr(xl^{k,t+1}|x_i^t,x_i^{t+1},a_i) \tag{3.7}$$

  Note that although such individual factors $xl^{k,t+1}$ can have intra-stage dependencies on other OLAFs $xl^{l,t+1}$ (i.e., as the functional form $\Pr(xl^{k,t+1}|x_i^t,x_i^{t+1},a_i)$ reveals, $xl^{k,t+1}$ can depend on x_i^{t+1} which can include other OLAFs $xl^{l,t+1}$), the product term $\Pr(xl_i^{t+1}|x_i^t,xn_i^{t+1},a_i)$ itself can only have intra-stage dependencies on xn_i^{t+1}; the intra-stage OLAFs xl_i^{t+1} will not appear in the conditioning set ('behind the pipe') as they have all been multiplied in (they are 'before the pipe'). As long as the 2DBN is non-cyclical this should not present any problems. [7]
- $\Pr(xn_i^{t+1}|x_i^t,xl_i^{t+1},y_i^t,y_i^{t+1},a_i,a_{\neq i})$ the product of NLAF probabilities:

$$\Pr(xn_i^{t+1}|x_i^t,xl_i^{t+1},y_i^t,y_i^{t+1},a_i,a_{\neq i}) = \prod_{k\in NLAF(i)} \Pr(xn^{k,t+1}|x_i^t,x_i^{t+1},y_i^t,y_i^{t+1},a_i,a_{\neq i}) \tag{3.8}$$

- $\Pr(y_i^{t+1}|x_i^t,x_i^{t+1},y_i^t,a_i,a_{\neq i})$ the product of probabilities of the NMFs y^k:

$$\Pr(y_i^{t+1}|x_i^t,x_i^{t+1},y_i^t,a_i,a_{\neq i}) = \prod_{k\in NMF(i)} \Pr(y^{k,t+1}|x_i^t,x_i^{t+1},y_i^t,y_i^{t+1},a_i,a_{\neq i}) \tag{3.9}$$

Value Function An LFM contains an fPOSG and as such best-response values for an agent i can be computed using the techniques discussed above in Section 3.1. In particular, we can just ignore the local state function and apply (3.1) with the previously stated definitions of $R_i(b_i^g,a_i)$ (3.2) and $\Pr(o_i^{t+1}|b_i^g,a_i)$ (3.3).

Clearly, however, we would like to now rewrite the value function in a way that represents the local structure imposed by the LFM requirements and exploits this for computational benefits. The former is possible: for an LFM, we can indeed derive a expression for $R_i(b_i^g,a_i)$ that is more local (see Appendix A.2.1).

$$R_i(b_i^g,a_i) = \sum_{x_i^t}\sum_{x_i^{t+1}} R_i(x_i^t,a_i,x_i^{t+1})\Pr(x_i^t,x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}), \tag{3.10}$$

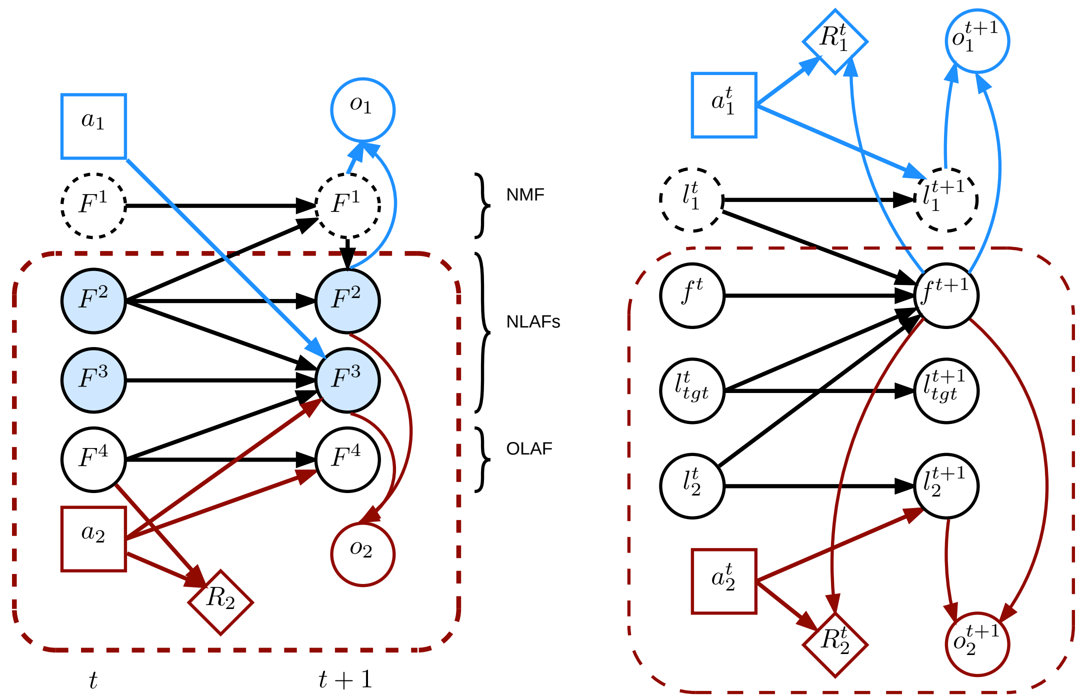
where (remember $s^t = \langle x_i^t,y_i^t\rangle$)

$$\Pr(x_i^t,x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}) \triangleq \sum_{y_i^t}\sum_{a_{\neq i}} \Pr(x_i^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}). \tag{3.11}$$

7. A more explicit way of writing this is as follows. In general the OLAFs can now depend on some NLAFs $xn_i^{ISD,t+1}$ that act as intra-stage dependencies:

$$\Pr(xl_i^{t+1}|x_i^t,xn_i^{ISD,t+1},a_i^t) \triangleq \prod_{k\in OLAF(i)} \Pr(xl^{k,t+1}|x_i^t,a_i^t,x^{ISD(k),t+1})$$

with $x^{ISD(k),t+1}$ denoting the intra-stage parents of $xl^{k,t+1}$. To reduce the notational burden, however, we will use the shorthands from (3.7).



(a) Illustration of an abstract local-form model for agent 2. Factors can be divided into non-modeled factors (F^1), non-locally-affected factors (F^2, F^3), and locally-affected factors (F^4). Also note that F^4 is reward-relevant, while F^2 and F^3 are observation-relevant factors.

(b) Local-form model for agent 2 in the housesearch problem without intra-strage dependencies.

Figure 5: Local-form models.

And, similarly, we can find a new, local, expression for the observation probability (Appendix A.2.2):

$$\Pr(o_i^{t+1}|b_i^g,a_i) = \sum_{x_i^{t+1}} \Pr(o_i^{t+1}|a_i,x_i^{t+1}) \Pr(x_i^{t+1}|b_i^g,a_i,\pi_{\neq i}) \tag{3.12}$$

where

$$\Pr(x_i^{t+1}|b_i^g,a_i) \triangleq \sum_{s^t} \sum_{a_{\neq i}} \Pr(x_i^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t \pi_{\neq i}) b_i^g(s^t,\vec{\theta}_{\neq i}^t). \tag{3.13}$$

These new definitions of $R_i(b_i^g,a_i)$, $\Pr(o_i^{t+1}|b_i^g,a_i)$ can be used directly in conjunction with (3.1).

However, even though these definitions (3.10) and (3.12) are local, they still depend on the global-form belief and this must perform summations over full states s^t and histories of other agents $\vec{\theta}_{\neq i}^t$ via (3.11) and (3.13), rendering them intractable for larger problems. In the next section, we will investigate formulations that are based on more local beliefs to try and overcome this computational hurdle.

4. Influence-Based Abstraction

In the previous section we introduced the GFBRM, which could be used to compute a best response against a fixed policy of other agents. This model gives a straightforward way of formulating the problem of computing a best-response. However, it is specified over the global state and internal state of other agents (i.e., their AOHs), which means that solving this model is computationally intractable.

To provide a more localized perspective, the local-form POSG defines for each agent a subset of factors that it should be concerned with. However, even if the policies of the other agents are fixed, it is not clear how an agent i can restrict its reasoning to its local state x_i: the non-modeled factors will still affect the local state transitions. Intuitively, we need to capture the influence that the non-modeled part of the problem exerts on the modeled part.

In this section, we formalize this intuition. In particular, we treat an LFM from the perspective of one agent and consider how that agent is affected by the other agents and can compute a best response against that 'incoming' influence.[8]

In an attempt to avoid notation overload, we first present a formulation without considering intra-stage connections. The general formulation that can deal with such connections is given in Section 5.

4.1 Definition of Influence

As discussed in Section 3.1, when the other agents are following a fixed policy, they can be regarded as part of the environment. The resulting decision problem can be represented by the complete unrolled DBN, as illustrated in Fig. 4. In this figure, a node F^t is a different node than F^{t+1} and an edge at (emerging from) stage t is a different from the edge at $t+1$ that corresponds to the same edge in the 2DBN. Given this uniqueness of nodes and edges, we can define the 'influence' as follows.

4.1.1 Influence Links, Sources & Destinations

Intuitively, the influence of other agents is the effect of those edges leading into the agent's local problem. We say that every directed edge from outside the local model (e.g., from an NMF or action of another agent) to inside the local model (e.g., to an modeled state factor, observation variable, or reward), is an *influence link* $\langle u^t,v^t \rangle$, where u^t is called the *influence source* and v^t is the *influence destination*. In this section, we will assume that influence links traverse a stage of the process (i.e., that the influence source for a destination v^t lies in the stage $t-1$), but since we will also consider

8. An agent also exerts 'outgoing' influence on other agents, but this is irrelevant for best response computation.

intra-stage influence links at a later point in this document, to keep notation consistent, we label an entire influence link with the stage-index of its destination.

For example, let's consider the House Search problem's LFM shown in Figure 5b on page 14. It shows that the link from l_1^t, the location of agent 1, to the 'target found' variable f^{t+1} is an influence link, such that we would write the link as $\langle u^{t+1} = l_1^t, v^{t+1} = f^{t+1} \rangle$, similarly $\langle u^t = l_1^{t-1}, v^t = f^t \rangle$ would denote the influence link in the preceding time step.

Assuming no intra-stage influence links, an influence source u^t can be either an action a_j^{t-1} or non-modeled state factor y^{t-1}. We write $u_i^t = \langle y_u^{t-1}, a_u^{t-1} \rangle$ for an instantiation of all influence sources exerting influence on agent i at stage t. That is, in the case of multiple influence links pointing to modeled factors in stage t, y_u^{t-1} denotes the (value of) influence sources that are state factors, while a_u^{t-1} corresponds to those influence sources that are actions. For instance, in our House Search example, $y_u^{t-1} = \{l_1^{t-1}\}$, while $a_u^{t-1} = \emptyset$ since there are no actions that are influence sources. We write $\vec{\theta}_u^{t-1}$ for the AOHs of those other agents whose action is an influence source (i.e., $\vec{\theta}_u^{t-1}$ and a_u^{t-1} involve the same agents) .

In general, an influence destination can be either a (per definition non-locally-affected) modeled factor xn^t, an observation variable o_i^t, or a local reward node R_i^t. But Definition (9) requires reward- or observation-relevant factors to be included in the local state; effectively we restrict ourselves to the setting where the influence destination is an NLAF. This restriction is without loss in generality: because we will introduce (in Section 5) the machinery to deal with *intra-stage* influence links, influences on observations and rewards can easily be dealt with by introducing a 'dummy' NLAF that acts as a proxy for the observation or reward.[9] A similar construction can be used to deal with settings where actions of other agents would directly influence the observations or rewards of the agent under concern. As such, the capability of dealing with such intra-stage dependencies is critical for the applicability of the theory of influence-based abstraction.

4.1.2 Sufficient Information to Predict Influences: D-Separating Sets

If agent i would in advance know the value of its influence sources at different time steps, it could easily compute its best response by making use of only this knowledge and its local model. Of course, this is in general not possible, since the influence sources are random variables. However, the influence exerted on agent i can be captured if we know the probability distribution over their values. That is, in order to predict the probability of some xn_i^{t+1} (i.e., an influence destination) agent i only cares about the following marginal probability

$$\sum_{u_i^{t+1}} \Pr(xn_i^{t+1} | x_i^t, a_i^t, u_i^{t+1}) \Pr(u_i^{t+1} | \ldots), \tag{4.1}$$

where the dots $(\ldots)$ indicate any information that agent i needs to predict the probability of the values of the influence sources as accurately as possible. Moreover, since these probabilities will be used to plan a best response, correlations between influence sources and local states are important. This unfortunately means that in general, we might need to condition $\Pr(u_i^{t+1}|...)$ on the entire history of actions, observations and and local states.

Fortunately, it turns out that in many cases we can find substantially more compact representations of the conditional probability of u_i^{t+1}, by making use of the concept of *d-separation* in graphical models [6, 19]. In particular, when two nodes A,B in a Bayesian network are d-separated given some of subsets D of evidence nodes, then A and B are conditionally independent given D, which means that $\Pr(A|D,B) = \Pr(A|D)$ and vice versa. Whether nodes are d-separated can be easily checked, by applying a small set of rules on the graph, for details please see, e.g., [6, chapter 8].

9. E.g., to deal with an observation destination, we can transform the observation o_i to a state factor F^o and introduce a new observation variable that has a deterministic CPT depending only on F^o.

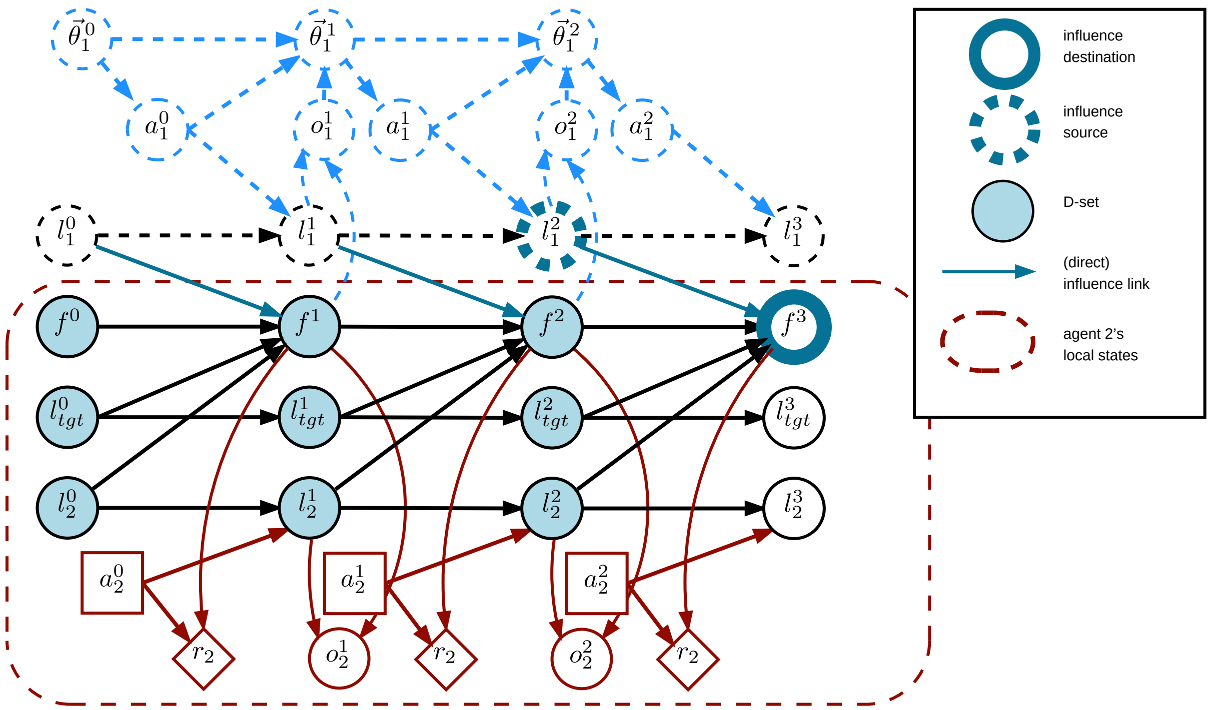


Figure 6: Illustration of the incoming influence on protagonist agent $i = 2$ in House Search at stage $t = 3$. f^3 is the only influence destination, with influence source $y_u^2 = l_1^2$ (i.e., $u_i^3 = \langle l_1^2 \rangle$). The shaded nodes indicate the d-separating set D_i^3, which, in accordance with (4.2), d-separates the influence source l_1^2, from agent 2's AOH $\vec{\theta}_i^t$ and possibly remaining x_i^t other local variables (in this case there are no such variables, but one could imagine adding a battery life variable for agent 2).

Now, we can define the influence as a conditional probability distribution over u_i^{t+1}, given a d-separating set. Specifically, let D_i^{t+1} be a subset of variables (possibly including state factors and actions) in the local problem of agent i at stages $0,\ldots,t$,

Definition 10 (D-separating set)**.** D_i^{t+1} is a *d-separating set for agent i's influence at stage* $t+1$ if and only if it d-separates $y_u^t,\vec{\theta}_u^t$ from $x_i^t,\vec{\theta}_i^t$. That is, if:

$$\forall_{y_u^t,\vec{\theta}_u^t} \qquad \Pr(y_u^t,\vec{\theta}_u^t | x_i^t,\vec{\theta}_i^t,D_i^{t+1},b^0,\pi_{\neq i}) = \Pr(y_u^t,\vec{\theta}_u^t | D_i^{t+1},b^0,\pi_{\neq i}). \tag{4.2}$$

This definition implies that remembering more than D_i^{t+1} is not useful for predicting $y_u^t,\vec{\theta}_u^t$ and hence for predicting $u_i^{t+1} = \langle y_u^t,a_u^t \rangle$ (given their policies, the actions of other agents only depend on their AOHs).

Example. Fig. 6 illustrates a d-separating set D_i^3 for agent $i = 2$ in House Search. It shows that, in order to accurately compute the probability of influence source l_1^2, agent 2 needs to condition on $f^{0:2}$, the history of the found variable, as well as the histories of the location of the target l_{tgt} and it own location l_2. This dependence on the history in general leads to large conditioning sets, but in many cases the history can be represented more compactly. For instance, in House Search the 'found' variable can only switch on (not off) which means that its history $f^{0:t}$ can be summarized compactly. And in cases where the target is static the same holds for $l_{tgt}^{0:t}$.

Example. Fig. 7 shows a similar diagram for a variant of the planetary exploration domain [45]. Here agent 2 is a mars rover which is tasked with navigating to some goal. Agent 1 is a satellite

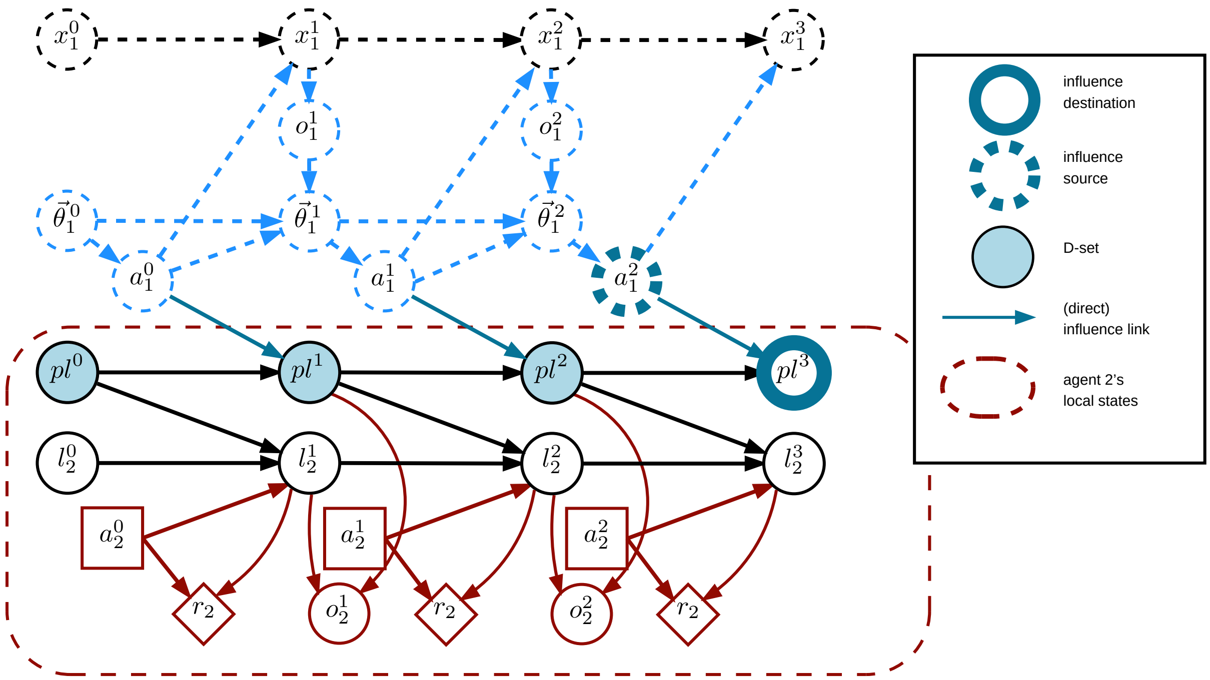


Figure 7: Illustration of the influence experienced by the mars rover (agent $i = 2$) at stage $t = 3$ in the PLANETARY EXPLORATION domain. If the satellite (agent 1) computes and transmits a plan (*pl*), the rover can more effectively navigate from that point onward.

which can aid the rover by planning a path, but this will use up computational resources and battery power (which it may want to use to support other rovers too, for instance). In the figure this is illustrated by the fact that the action of agent 1 $a_1 \in \{NOOP,PLAN\}$ (which now is the influence source) determines if there is a plan available for agent 2, modeled by a binary variable *pl* (which is the influence destination). In this example, the d-separating set only contains this variable *pl*. Again its history can be compactly summarized: as having the plan can only turn true, we can just store how long ago *pl* was switched to true.

4.1.3 The Influence Exerted on Agent i

Given the above machinery, we can now state our definition of influence:

Definition 11 (Incoming Influence)**.** The *incoming influence* at stage $t+1$, denoted $I^{t+1}_{\rightarrow i}(\pi_{\neq i})$, is a conditional probability distribution over values of the influence sources:

$$I(u_i^{t+1}|D_i^{t+1}) \triangleq \sum_{\vec{\theta}_u^t} \Pr(a_u^t|\vec{\theta}_u^t) \Pr(y_u^t,\vec{\theta}_u^t|D_i^{t+1},b^0,\pi_{\neq i}). \tag{4.3}$$

Note that, to reduce notational burden we drop arguments that can be inferred, such as b^0,$\pi_{\neq i}$. That is, $I(u_i^{t+1}|D_i^{t+1})$ is shorthand for $I^{t+1}_{\rightarrow i}(u_i^{t+1}|D_i^{t+1},b^0,\pi_{\neq i})$. In cases where we want to refer to this distribution as a whole, we will write $I^{t+1}_{\rightarrow i}(\pi_{\neq i})$.

We will also say that this is the influence *exerted* on agent i at stage t or *experienced* by agent i at stage $t+1$. So far, these notions coincide, but when we consider intra-stage connections in the next section, we will discriminate between these concepts.

Finally, we are in position to specify the complete influence on agent i:

Definition 12. An *incoming influence point* $I_{\rightarrow i}(\pi_{\neq i})$ for agent i, specifies the incoming influences for all stages $I_{\rightarrow i}(\pi_{\neq i}) = \left(I^1_{\rightarrow i}(\pi_{\neq i}), \ldots, I^h_{\rightarrow i}(\pi_{\neq i})\right)$.

As we will see in the remainder of this paper, an influence point contains all the information about the non-modeled part of the problem that agent i needs to compute a best response 'locally', i.e., only using its local model and that influence point. This can bring computational benefits for instance when there would be changes in the local model that require repeatedly performing planning, or in cases where the influence point can be computed easily. This form of influence-based abstraction, however, is not providing a free lunch [50]: in general computing the incoming influences (4.3) for the different stages comprise a set of challenging inference problems. However, many special cases of problems have been identified in the past [4, 3, 41, 36, 44, 45, 46, 42, 48, 47, 33], and IBA gives a unified perspective on these. Moreover, it can be used as tool to identify further special cases that allow for efficient solution, such as the class of ND-POMDPs discussed in [33]. Given the potential benefits of using influence representations [47], such future search for special cases of problems that allow for compact influence specifications together with the inference algorithms that efficiently compute these is an important line of research. Our definition of influence in this section provides the general framework in which these special cases should be sought.

4.2 The Influence-Augmented Local Model (IALM)

Given the above definition of influence, we can now define a smaller *local* model for our protagonist agent i. The main idea is that given an incoming influence point, agent i no longer needs to reason over the non-modeled part of the problem. Instead, it can use the influence to compute marginal probabilities as expressed by (4.1), and this will allow it to compute an exact best-response.

In this section, we will first investigate a single NLAF and how the influence on it can be incorporated. Then we move to talk about the case where multiple variables in the local state $S(i)$ are non-locally affected. Then we proceed to the formal definition of the IALM, and how it can be solved.

4.2.1 Induced CPTs

In the case of a single influence destination, we can interpret (4.1) as constructing a new 'influence-induced' CPT:

Definition 13 (Induced CPT)**.** Let xn^{t+1} be an influence destination, and u^{t+1} (the instantiation of) the corresponding influence sources. Given the influence $I^{t+1}_{\rightarrow i}(\pi_{\neq i})$, and its d-separating set D^{t+1}_i, we define the *induced CPT* for xn^{t+1} as the CPT that has probabilities:

$$p_{I^{t+1}_{\rightarrow i}}(xn^{t+1}|x^t_i,D^{t+1}_i,a_i) = \sum_{u^{t+1}_i=\langle y^t_u,a_u\rangle} \Pr(xn^{t+1}|x^t_i,a_i,u^{t+1}_i) I(u^{t+1}_i|D^{t+1}_i) \tag{4.4}$$

It is important to note that an induced CPT is specified purely in *local* terms, i.e., making use of variables that are modeled by our protagonist agent i. Therefore, the basic idea is that we can now define a smaller *local* model—which we will call the *Influence-Augmented Local Model (IALM)*—by replacing the CPTs of influence destinations (i.e., NLAFs) by induced CPTs.

4.2.2 Dealing With Multiple NLAFs

However, in case that there are multiple NLAFs, i.e., multiple variables xn^{t+1} in the local state space $S(i)$ that are affected nonlocally at the same stage $t+1$, the story is slightly more involved, since we need to deal with their correlations.

Ideally, we would want to treat induced CPTs in the same way as normal CPT; that is, we would represent the joint probability of NLAFs as a the product of induced CPTs:

$$\Pr(xn_i^{t+1}|x_i^t,D_i^{t+1},a_i,I_{\to i}^{t+1}) = \prod_{k\in NLAF(i)} p_{I_{\to i}^{t+1}}(xn^{k,t+1}|x_i^t,D_i^{t+1},a_i). \tag{4.5}$$

However, in general this is not possible since the different $xn^{k,t+1}$ are correlated via any common influence sources. That is, in general the probability is given by:

$$\Pr(xn_i^{t+1}|x_i^t,D_i^{t+1},a_i,I_{\to i}^{t+1}) = \sum_{u_i^{t+1}=\langle y_u^t,a_u\rangle} I(u_i^{t+1}|D_i^{t+1}) \prod_{k\in NLAF(i)} \Pr(xn^{k,t+1}|x_i^t,a_i,u_i^{t+1}) \tag{4.6}$$

Of course, in certain cases a factorization as induced CPTs is possible. The above equations directly make clear when this is the case.

Proposition 1. *If each NLAF $xn^{k,t+1}$ has its own influence sources $u^{k,t+1}$ (and these do not overlap), and if these sources are conditionally independent given D_i^{t+1}:*

$$I(u_i^{t+1}|D_i^{t+1}) = \prod_{k\in NLAF(i)} I(u^{k,t+1}|D_i^{t+1}),$$

then the joint probability of NLAFs factorizes as a the product of induced CPTs.

Proof. Under stated conditions, we can rewrite:

$$\begin{aligned}
(4.6) &= \sum_{u_i^{t+1}=\langle\dots,u^{k,t+1},\dots\rangle} I(u_i^{t+1}|D_i^{t+1}) \prod_{k\in NLAF(i)} \Pr(xn^{k,t+1}|x_i^t,a_i,u^{k,t+1}) \\
&= \sum_{u_i^{t+1}=\langle\dots,u^{k,t+1},\dots\rangle} \left[\prod_{k\in NLAF(i)} I(u^{k,t+1}|D_i^{t+1})\right] \prod_{k\in NLAF(i)} \Pr(xn^{k,t+1}|x_i^t,a_i,u^{k,t+1}) \\
&= \sum_{u_i^{t+1}=\langle\dots,u^{k,t+1},\dots\rangle} \prod_{k\in NLAF(i)} I(u^{k,t+1}|D_i^{t+1}) \Pr(xn^{k,t+1}|x_i^t,a_i,u^{k,t+1}) \\
&= \prod_{k\in NLAF(i)} \sum_{u^{k,t+1}} I(u^{k,t+1}|D_i^{t+1}) \Pr(xn^{k,t+1}|x_i^t,a_i,u^{k,t+1}) = (4.5)
\end{aligned}$$

□

4.2.3 The IALM: A Formal Model to Incorporate Influence

Here we formally define the IALM, which is a non-stationary POMDP, since at every stage the influence destinations can be influenced in a different manner.

Definition 14 (IALM)**.** Given an LFM, $\mathcal{M}^{LFM}$, and profile of policies for other agents $\pi_{\neq i}$, an *Influence-Augmented Local Model (IALM)* for agent i is a POMDP $\mathcal{M}_i^{IALM}(\mathcal{M}^{LFM},\pi_{\neq i}) = \left\langle \bar{\mathcal{S}},\mathcal{A}_i,\bar{T}_i,\bar{R}_i,\mathcal{O}_i,\bar{O}_i,h,b_i^{l,0}\right\rangle$, where

- $\bar{\mathcal{S}}$ is the set of augmented states $\bar{s}_i^t = \langle x_i^t,D_i^{t+1}\rangle$ that specify an underlying local state of the POSG, as well as the d-separating set D_i^{t+1} for the next-stage influences. Note that D_i^{t+1} very typically needs to include certain state factors for stage t, such that x_i^t and D_i^{t+1} both will specify such variables. This is no problem, as long as they specify consistent assignments; we define $\bar{\mathcal{S}}$ to be the set of states that are consistent.
- $\mathcal{A}_i,\mathcal{O}_i$ are the (unmodified) sets of actions and observations for agent i.
- The transition function $\bar{T}_i(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t)$ on which we will elaborate below.

- The observation function $\bar{O}_i(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1}) = O(o_i^{t+1}|a_i^t,x_i^{t+1})$, since agent i's observations only depend on its local state (cf. Definition 9, property 1)
- The reward function $\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1}) = R_i(x_i^t,a_i^t,x_i^{t+1})$, since agent i's rewards only depend on its local state (cf. Definition 9, property 2).
- h is the unmodified horizon.
- $b_i^{l,0}$ is the initial state distribution, which is a *local-form belief.* It is a distribution over augmented states $\bar{s}_i^0 = \langle x_i^0,D_i^1\rangle$. Since for the first stage D_i^1 can only contain elements from x_i^0, it can trivially be constructed from a probability distribution over x_i^0, and such a distribution can be constructed from b^0, as we discuss in a bit more detail below.

In defining $\bar{T}_i$ and $b_i^{l,0}$, a few subtleties arise that we now discuss.

Transition Probabilities Clearly, the IALM's transition probabilities should express

$$\bar{T}_i(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t) \triangleq \Pr(\langle x_i^{t+1},D_i^{t+2}\rangle|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\to i}^{t+1}).$$

For such probabilities to be specified, we need some further requirements on the d-separating sets. In particular, we require that (the instantiation of) D_i^{t+2} is fully specified by x_i^t,a_i^t,x_i^{t+1} and D_i^{t+1} such that we can write D_i^{t+2} as some function d of those arguments: $D_i^{t+2} = d(x_i^t,a_i^t,x_i^{t+1},D_i^{t+1})$. If this is the case, we can write:

$$\Pr(\langle x_i^{t+1},D_i^{t+2}\rangle|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\to i}^{t+1}) = \Pr(x_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\to i}^{t+1})\mathbf{1}_{\{D_i^{t+2},d(x_i^t,a_i^t,x_i^{t+1},D_i^{t+1})\}},$$

where $\mathbf{1}_{\{\cdot,\cdot\}}$ denotes the Kronecker delta function.

A typical way to fulfill the requirement that D_i^{t+2} is fully specified by x_i^t,a_i^t,x_i^{t+1} and D_i^{t+1} is to assume that the d-separating sets for all stages are chosen as the history of the same subset $D_i \subseteq S(i)$ of modeled features.

Example. Looking at Figure 6 on page 17, the d-separating set D_2^3 for predicting f^3 is given by the history of the 'found', 'location of target' and 'location of agent 2' variables. So we can write $D_2 = \{f,l_{tgt},l_2\}$, and define D_2^3 to be its history at stage $t = 2$: $D_2^3 = \vec{D}_2^2$.

The probabilities $\Pr(x_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t)$ now are factored as the product of the CPTs of the OLAFs and the induced probabilities for the NLAFs:

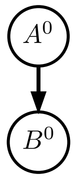
$$\begin{aligned}\bar{T}_i(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t) &\triangleq \Pr(x_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\to i}^{t+1})\mathbf{1}_{\{D_i^{t+2},d(x_i^t,a_i^t,x_i^{t+1},D_i^{t+1})\}},\\ &= \Pr(xn_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\to i}^{t+1})\Pr(xl_i^{t+1}|x_i^t,xn_i^{t+1},a_i).\end{aligned} \tag{4.7}$$

Here the first term is given by (4.6) and the second term is given by (3.7).[10]

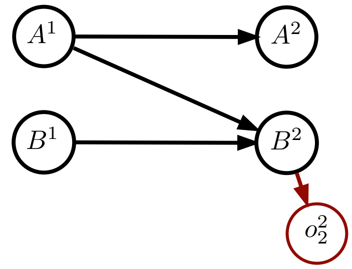
Initial Local State Distribution Here we discuss some of the issues involved in defining the initial belief in the IALM. Note that in a factored models such as fPOSGs, the initial state distribution b^0 is specified as a Bayesian network G^0. Together with the 2DBN, $G^{\to}$ (which in fact is a conditional probability distribution), it can form the unrolled DBN $G = \text{unroll}(G^0,G^{\to})$ which specifies the joint distribution over all the state variables, as is illustrated in Fig. 8. Note that the figure gives a simplified representation not involving any actions.

Now we will discuss how to specify the initial belief $b_i^{l,0}(x_i^0)$ of the IALM, the basic idea is to simply restrict G^0 to those variables in the set $S(i)$ of agent i's local state variables. However, this can lead to problems when there are arrows in G^0 pointing from variables not included in $S(i)$ to variables

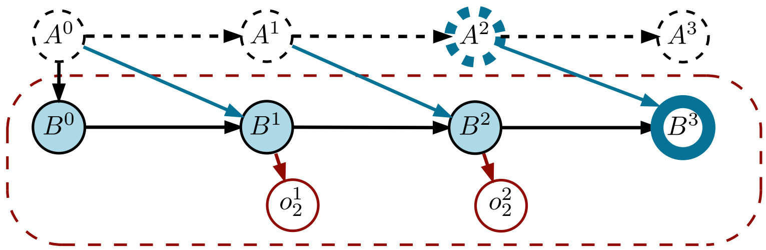
10. Note that, even though we have not dealt with intra-stage dependencies (ISDs) in the description of influences in this section, we refer back to the term $\Pr(xl_i^{t+1}|x_i^t,xn_i^{t+1},a_i)$ from section 3 which does allow for ISDs from NLAFs to OLAFs. This will allow us to make only minimal changes to the definition of $\bar{T}_i$ when we do deal with ISDs in Section 5.



(a) The Bayesian network G^0 representing the initial belief.



(b) The 2DBN (a conditional Bayesian network) $G^{\rightarrow}$ representing the transition and observation probabilities.



(c) The unrolled network $G = \text{unroll}(G^0, G^{\rightarrow})$. To convert it to an IALM, the local-form initial belief $b_i^{l,0}(B^0)$ and incoming influences $I_{\rightarrow i}^{t+1}(A^t|B^0,\ldots,B^t)$ need to be computed via inference. See text for further explanation.

Figure 8: Construction of the the IALM.

included in $S(i)$. For instance, in Fig. 8, the initial belief is factored: $b^0(s) = \Pr(A^0)\Pr(B^0|A^0)$. The initial local-form belief, however, should only be specified over B^0. The solution is to marginalize out the dependencies:

$$b_i^{l,0}(B^0) = \sum_{A^0} \Pr(A^0)\Pr(B^0|A^0).$$

This is also gives the general recipe for any other problem: construction of $b_i^{l,0}$ from b^0 is a marginal inference task. Certainly, for certain complex problems this could be intractable, but the hope is that for many real-world problems the prior b^0 is sufficiently sparsely structured for this not to be an issue. Also, any of the vast number of (exact or approximate) inference methods developed in the last decades can be used [19, 8, 16, 23, 43].

Impact of Correlations of Initial State Factors on The D-separating Set Note that the correlation of the initial state distribution can affect d-separation and therefore what variables need to be included in the d-separating set D_i^t. For instance, if in the above example there additionally is a state factor C, which is not connected to A or B in the 2DBN $G^{\rightarrow}$, but which is a parent of A in G^0, we get the unrolled DBN as shown in Fig. 9.

Now, to define the IALM, we will need to induced probability of B^3, which according to (4.6) can be written as

$$\Pr(B^3|\left\langle B^2, D_i^3\right\rangle, I_{\rightarrow i}^3) = \sum_{A^2} I(A^2|D_i^3)\Pr(B^3|B^2, A^2).$$

Therefore D_i^3 needs to contain any variables that can be used to better predict A^2 (more formally, any variables that d-separate A^2 from $\vec{\theta}_i^t$ and any remaining variables x_i^t, cf. Definition 10). However, looking at the figure, we see that that means that C^0 needs to be included in the d-set.

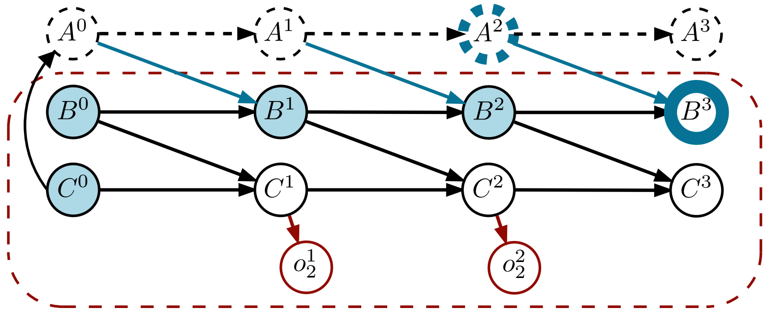


Figure 9: Impact of initial belief connectivity on the d-separating set of the IALM.

At the same time, however, we see that we do not need to condition on the entire history $\vec{C}^t$. This may apear counter intuitive, since observations at later time-steps (e.g, o_i^2) which depend on $\vec{C}^t$ certainly provide information about A^2. But this is precisely the point: by including C^0 in the d-separating set, it becomes part of the hidden state $\bar{s}_i^2 = \langle x_i^2, D_i^3 \rangle = \langle \langle C^2 \rangle, \langle B^0, B^1, B^2, C^0 \rangle \rangle$, and the later observations certainly provide information as to what that hidden state is.

Note. We note that this discussion neatly exemplifies some different types of structure we can expect to encounter when dealing with abstraction in structured decision making processes. This is important also in the area of deep learning where much of the representations are learned automatically, since no learning methods are effective without the appropriate inductive biases [22, 49]. For instance, convolutional neural networks are so successful for image processing because they exploit the fact that there is local and repeated structure in real-world images. In a similar way, we are noticing here that certain forms of structure, such as dependence on certain state factors at stage $t = 0$, might be common in sequential decision processes involving abstraction, since we identify problem structure which would lead to such dependencies. Recent research provides preliminary evidence that structure as implied by influence-based abstraction can be effectively uses to bias deep reinforcement learning [10].

4.3 Planning in an IALM

Here we look at how we can reason using an IALM. It turns out that this is surprisingly simple, since an IALM *is a* (special case of) POMDP.

Observation. An influence-augmented local model is a POMDP.

Proof. This can simply be verified by comparing Definition 14 to the definition of a POMDP (Definition 1). □

This means that belief updates and definition of value functions follow as usual. For completeness and future reference, we write these out in detail below.

4.3.1 Local-form Belief Update

As implied by Definition 14, in an IALM, an agent uses a *local-form belief*:

Definition 15 (local-form belief)**.** A *local-form belief* $b_i^{l,t}$ for an IALM constructed for agent i is the posterior probability distribution over augmented states $\bar{s}_i^t = \langle x_i^t, D_i^{t+1} \rangle$.

The belief update for such a local-form belief is as in a regular POMDP, cf. (2.1):

$$BU(b_i^l,a_i^t,o_i^{t+1})(\bar{s}_i^{t+1}) = \frac{1}{\Pr(o_i^{t+1}|b_i^l,a_i^t)}\bar{O}_i(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1})\sum_{\bar{s}_i^t}\bar{T}_i(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t)b_i^l(\bar{s}_i^t) =$$

$$\frac{1}{\Pr(o_i^{t+1}|b_i^l,a_i^t)}O(o_i^{t+1}|a_i^t,x_i^{t+1})\sum_{x_i^t,D_i^{t+1}}\Pr(x_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\rightarrow i}^{t+1})\mathbf{1}_{\{D_i^{t+2},d(x_i^t,a_i^t,x_i^{t+1},D_i^{t+1})\}}b_i^l(x_i^t,D_i^{t+1}) \tag{4.8}$$

The expected observation probability (the normalization factor) in this case is given by (see Appendix A.3.2)

$$\begin{aligned}\Pr(o_i^{t+1}|b_i^l,a_i^t) &= \mathbf{E}_{\bar{s}_i^t\sim b_i^l,\bar{s}_i^{t+1}\sim\bar{T}(\bar{s}_i^t,a_i,\cdot)}\left[\bar{O}(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1})\right]\\ &= \mathbf{E}_{\langle x_i^t,D_i^{t+1}\rangle\sim b_i^l,\langle x_i^{t+1},D_i^{t+2}\rangle\sim\bar{T}(\langle x_i^t,D_i^{t+1}\rangle,a_i,\cdot)}\left[O(o_i^{t+1}|a_i^t,x_i^{t+1})\right]\\ &= \sum_{x_i^{t+1}}O(o_i^{t+1}|a_i^t,x_i^{t+1})\Pr(x_i^{t+1}|b_i^l,a_i^t,I_{\rightarrow i}^{t+1}),\end{aligned} \tag{4.9}$$

with

$$\Pr(x_i^{t+1}|b_i^l,a_i,I_{\rightarrow i}^{t+1}) \triangleq \sum_{x_i^t,D_i^{t+1}}\Pr(x_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\rightarrow i}^{t+1})b_i^l(x_i^t,D_i^{t+1}). \tag{4.10}$$

4.3.2 IALM Value

Putting everything togetting, we can show that for an IALM, the value function is similar to the normal POMDP value function:

Proposition 2 (IALM value function)**.** *The value function is given by*

$$Q_i(b_i^l,a_i^t) = R_i(b_i^l,a_i^t) + \gamma\sum_{o_i^{t+1}}\Pr(o_i^{t+1}|b_i^l,a_i^t)V_i(BU(b_i^l,a_i^t,o_i^{t+1})),$$

$$V_i(b_i^l) = \max_{a_i}Q_i(b_i^l,a_i),$$

where

$$\begin{aligned}R_i(b_i^l,a_i^t) &= \mathbf{E}_{\bar{s}_i^t\sim b_i^l,\bar{s}_i^{t+1}\sim\bar{T}(\bar{s}_i^t,a_i^t,\cdot)}\left[\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1})\right]\\ &= \sum_{x_i^t}\sum_{x_i^{t+1}}R_i(x_i^t,a_i^t,x_i^{t+1})\Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I_{\rightarrow i}^{t+1})\end{aligned} \tag{4.11}$$

with

$$\Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I_{\rightarrow i}^{t+1}) \triangleq \sum_{D_i^{t+1}}\Pr(x_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\rightarrow i}^{t+1})b_i^l(x_i^t,D_i^{t+1}). \tag{4.12}$$

Proof. This follows from the value function of regular POMDPs together with the derivations of $R_i(b_i^l,a_i^t)$ and $\Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I_{\rightarrow i}^{t+1})$ in Appendix A.3.1. ∎

The solution of the IALM gives the influence-based best-response value, defined as the value of the initial local-form belief:

$$V_i(I_{\rightarrow i}(\pi_{\neq i})) \triangleq V_i(b_i^{l,0}). \tag{4.13}$$

5. IBA With Intra-Stage Dependencies

In the previous section, we assumed that all influence links span a time step, but in general intra-stage connections can be useful to specify a more intuitive model, as for House Search in Fig. 2a. Moreover intra-stage connections enable us to introduce 'dummy' variables, as discussed in 4.1.1. Without this capability, the restriction of including all observation-relevant and reward-relevant variables in the local state would limit the applicability of influence-based abstraction.

This means that we need to be able to deal with intra-stage influence sources, as illustrated in Fig. 5a. Therefore, this section extends our definition of influence to also be applicable for models that have such *intra-stage dependencies (ISDs).*

5.1 Definition of Influence under ISDs

5.1.1 Intra-stage Influence Sources

In settings with intra-stage dependencies, there is at least one non-modeled factor y^{t+1} that influences an NLAF xn^{t+1}. If there are multiple such factors, we let y_u^{t+1} denote them. Therefore, in order to perform IBA in settings with ISDs, we will need to predict influence sources $u_i^{t+1} = \langle y_u^t, a_u^t, y_u^{t+1} \rangle$. In order to correctly deal with the intra-stage sources y_u^{t+1}, we will additionally need to consider those variables that influence *them.*

Indirect Sources In particular, we use 'v' as the symbol to denote such 'indirect' or 'second order' influences and will write $x_v^t, y_v^t, a_i^t, a_v^t, x_v^{t+1}$ and y_v^{t+1} for the possible[11] ancestors in the 2DBN of intra-stage sources y_u^{t+1}. Now, we want to consider the probability of such y_u^{t+1}, and in general it is given by:

$$\Pr(y_u^{t+1} | x_v^t, y_v^t, a_i^t, a_v^t, x_v^{t+1}) = \sum_{y_v^{t+1}} \Pr(y_u^{t+1}, y_v^{t+1} | x_v^t, y_v^t, a_i^t, a_v^t, x_v^{t+1}), \tag{5.1}$$

with:

- $\Pr(y_u^{t+1}, y_v^{t+1} | x_v^t, y_v^t, a_i^t, a_v^t, x_v^{t+1})$ the product of CPTs of (both direct and indirect) intra-stage sources,
- x_v^t are those state factors in at stage t ("in the left-hand slice of the 2DBN") that are modeled by agent i, and are ancestor to an intra-stage influence source of agent i at stage $t+1$ ("in the right-hand slice of the 2DBN"),
- y_v^t are those state factors in the left-hand slice of the 2DBN that are not modeled by agent i, but are ancestor to an influence destination of agent i,
- x_v^{t+1}, y_v^{t+1} are the modeled respectively unmodeled state factors at state $t+1$ that are ancestors to an intra-stage influence source,
- a_i^t might directly or indirectly affect an an intra-stage influence source, in which case it needs to be included in (5.1),
- a_v^t are the actions of other agents that somehow are ancestors of an intra-stage influence source.

We will also write $\vec{\theta}_v^t$ for the AOHs of the agents v that correspond to a_v^t (i.e., those agents of which the action is an ancestor in the 2DBN of an influence destination of agent i).

All sources So far we have introduced notation using u for direct sources and using v for indirect sources. We will also want to consider the union of direct and indirect sources, and for these purposes we will write w. For example, we will write $a_w^t = \langle a_u^t, a_v^t \rangle$ for the actions of agents that either directly or indirectly influence an influence destination.

11. Of course, in any given problem not all of these types of variables are relevant. For instance, if there is no action a_j^t of another agent j that would influence an ISD influence source , then a_v^t can be removed from the equations.

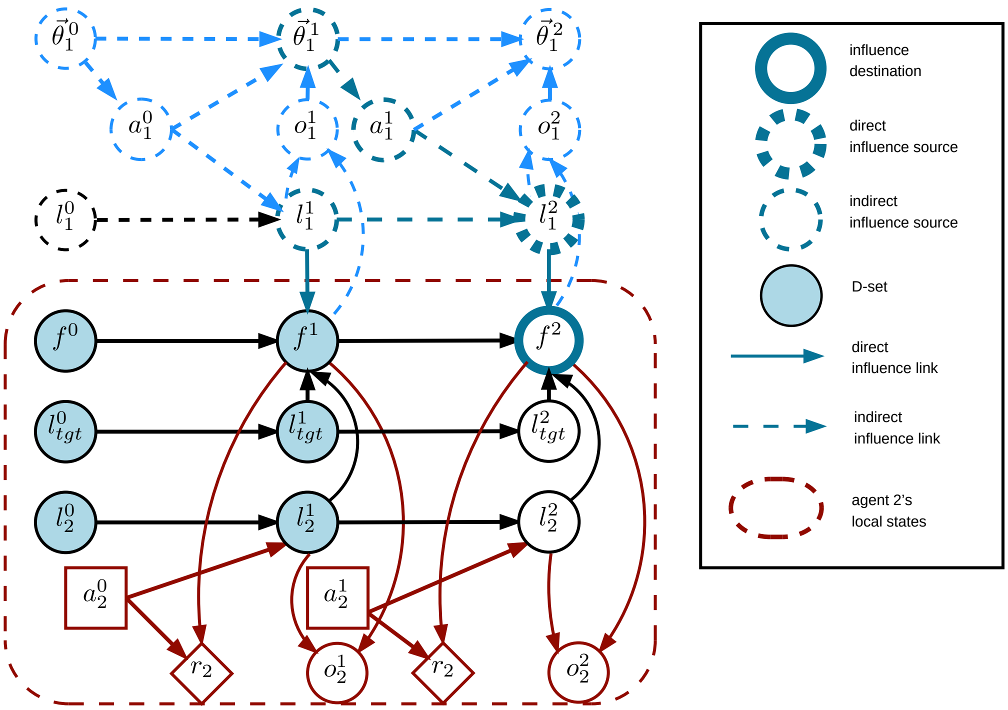


Figure 10: Illustration of the influence experienced by protagonist agent $i = 2$ in the intra-stage version of House Search at stage $t = 2$. f^2 is the influence destination, with *direct* influence source $y_u^2 = \langle l_1^2 \rangle$ (i.e., $u_i^2 = \langle l_1^2 \rangle$). Additionally, the figure highlights the *indirect* influence sources $y_v^1 = \langle l_1^1 \rangle$, $a_v^t = \langle a_1^1 \rangle$ and $\vec{\theta}_v^1 = \left\langle \vec{\theta}_1^1 \right\rangle$, which determine the influence that is exerted at stage $t = 1$. (Note that l_1^1 in fact is also a *direct* influence source for the influence *experienced at* stage $t = 1$.)

Example Fig. 10 illustrates the direct and indirect influence sources for HOUSE SEARCH. In order to be able to make accurate predictions of the influence destionation f^2 at stage $t=1$, we should be able to predict $y_v^1 = l_1^1, a_v^t = a_1^1$ as accurately as possible. Given that we assume access to the policy of agent 1, we can equivalently predict $y_v^1 = \langle l_1^1 \rangle, \vec{\theta}_v^1 = \langle a_1^1 \rangle$.

5.1.2 The D-Separating Set

We now build on this insight to define the d-separating set in problems with intra-stage dependencies:

Definition 16. The *d-separating set for agent i*, D_i, is a subset of variables (state factors and/or actions), such that the history of these variables d-separates $y_w^t, \vec{\theta}_w^t$ from $x_i^t, \vec{\theta}_i^t$. I.e., it is defined in such a way that

$$\forall_{y_w^t, \vec{\theta}_w^t} \qquad \Pr(y_w^t, \vec{\theta}_w^t | x_i^t, \vec{\theta}_i^t, D_i^{t+1}, b^0, \pi_{\neq i}) = \Pr(y_w^t, \vec{\theta}_w^t | D_i^{t+1}, b^0, \pi_{\neq i}). \tag{5.2}$$

As before this should be interpreted to mean: D_i^{t+1} d-separates $y_w^t, \vec{\theta}_w^t$ from those parts of $x_i^t, \vec{\theta}_i^t$ (i.e, of the local model) not contained in D_i^{t+1}.

Comparing Definition 16 with the earlier Definition 10, we see they are pleasingly similar; all that changed is that u's have been replaced with w' to now take into account the possiblity of indirect sources.

5.1.3 Definition of Influence under ISDs

With this as background, we are now in position to define the concept of influence in all its generality:

Definition 17 (Experienced Influence under ISDs)**.** The *influence experienced by agent i at stage $t+1$* is a conditional probability distribution over the direct influence sources:

$$\begin{aligned} I(u_i^{t+1} | D_i^{t+1}, x_v^t, a_i^t, x_v^{t+1}) &\triangleq \Pr(\langle y_u^t, a_u^t, y_u^{t+1} \rangle | D_i^{t+1}, x_v^t, a_i^t, x_v^{t+1}, b^0, \pi_{\neq i}) \\ &= \sum_{u_i^{t+1} = \langle y_v^t, a_v^t, y_v^{t+1} \rangle} \Pr(y_u^{t+1}, y_v^{t+1} | x_v^t, y_v^t, a_i^t, a_v^t, x_v^{t+1}) \sum_{\vec{\theta}_w^t} \pi_w(a_w^t | \vec{\theta}_w^t) \Pr(y_w^t, \vec{\theta}_w^t | D_i^{t+1}, b^0, \pi_{\neq i}) \end{aligned} \tag{5.3}$$

where

- u denote (direct) influence sources;
- v denote the 'second order' sources;
- w (as above) denotes the union of u and v;
- $\Pr(y_u^{t+1}, y_v^{t+1} | x_v^t, y_v^t, a_i^t, a_v^t, x_v^{t+1})$ is the term necessary the predict the intra-stage sources. It is a term that consists of the product of CPTs;
- $\pi_w(a_w^t | \vec{\theta}_w^t) = \prod_{i \in w} \pi_i(a_i^t | \vec{\theta}_i^t) = \pi_u(a_u^t | \vec{\theta}_u^t) \pi_v(a_v^t | \vec{\theta}_v^t)$ is the product of action probabilities according to the policies of the other agents that are relevant directly (the u) or indirectly for intra-stage sources (the v);
- $\Pr(y_w^t, \vec{\theta}_w^t | D_i^{t+1}, b^0, \pi_{\neq i}) = \Pr(y_u^t, y_v^t, \vec{\theta}_u^t, \vec{\theta}_v^t | D_i^{t+1}, b^0, \pi_{\neq i})$ predicts the non-modeled factors that are relevant directly (the u) or indirectly for intra-stage sources (the v), as well as the histories for the relevant agents.

We use $I_{\to i}^{t+1}(\pi_{\neq i})$ to denote the conditional distribution $I(\cdot | D_i^{t+1}, x_i^t, a_i^t, x_i^{t+1})$.

We make a few observations:

- The term $\Pr(y_u^{t+1}, y_v^{t+1} | x_v^t, y_v^t, a_i^t, a_v^t, x_v^{t+1})$ can be simplified as given by (5.1), but it is important to keep in mind that this resulting term requires actual inference and is not the product of CPTs anymore.

- Note that, in many cases, we will consider other agents that use deterministic policies, however, we chose to give the more general description that also allows for stochastic policies. In case of deterministic policies, the summation over a_v^t can be omitted, $a_{v/w}^t$ can be replaced by $\delta_{v/w}^t(\vec{o}_{v/w}^t)$, and $\vec{\theta}_w^t$ becomes $\vec{o}_w^t$.
- The dependence of $I(u_i^{t+1}|D_i^{t+1},x_v^t,a_i^t,x_v^{t+1})$ on a_i^t is only needed when a_i^t is an indirect source (i.e., it is an ancestor of y_u^{t+1} or y_v^{t+1}).

5.1.4 Exerted vs. Experienced Influence

Here we make a reinterpretation of the experienced influence at stage t+1 as the result of the influence exterted at stage t plus the effect of the intra-stage effects. While this does not fundamentally change anything about the definition of influence per Definition 17, it may provide some insight on the nature with which influence manifests itself in settings with intra-stage connections.

In particular, it is possible to define a distribution, only in terms of variables at stage t, which acts as a sufficient statistic to predict the intra-stage source. The intuition is that the *experienced influence,* can be thought of as being induced by the *exerted influence:*

- **Exerted Influence (at stage t):**

$$\Pr(y_w^t,a_w^t|D_i^{t+1},b^0,\pi_{\neq i}) = \Pr(y_u^t,y_v^t,a_u^t,a_v^t|D_i^{t+1},b^0,\pi_{\neq i}) = \sum_{\vec{\theta}_w^t} \pi_w(a_w^t|\vec{\theta}_w^t) \Pr(y_w^t,\vec{\theta}_w^t|D_i^{t+1},b^0,\pi_{\neq i}) \tag{5.4}$$

- **Experienced Influence (at $t+1$):**

$$\begin{aligned} I(u_i^{t+1}|D_i^{t+1},x_v^t,a_i^t,x_v^{t+1}) &= \Pr(y_u^t,a_u^t,y_u^{t+1}|D_i^{t+1},x_v^t,x_v^{t+1},b^0,\pi_{\neq i}) \\ &= \sum_{\langle y_v^t,a_v^t,y_v^{t+1}\rangle} \Pr(y_u^{t+1},y_v^{t+1}|x_v^t,y_v^t,a_i^t,a_v^t,x_v^{t+1}) \Pr(y_w^t,a_w^t|D_i^{t+1},b^0,\pi_{\neq i}) \end{aligned} \tag{5.5}$$

This last equation (5.5) clearly demonstrates how the experienced influence is induced by the exerted influence. The notion of exerted influence (5.4) lays a clear link to IBA in settings without ISDs (cf. Equation 4.3) and is conceptually useful since it isolates which information needs to be retained for each stage t. As such, we expect that any practical implementations for computing the influence by means of filtering (belief tracking) would use this as the primary quantity of interest.

5.2 Influence-Augmented Local Model (IALM)

Here we define the influence-augmented local model under intra-stage connections. Looking at Definition 14, we can conclude that the only changes that we need to make involve the transition function (4.7), repeated here for convience:

$$\bar{T}_i(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t) \quad = \quad \Pr(xn_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\rightarrow i}^{t+1}) \prod_{k\in OLAF(i)} \Pr(xl^{k,t+1}|x_i^t,a_i^t).$$

with

$$(4.6) = \Pr(xn_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\rightarrow i}^{t+1}) = \sum_{u_i^{t+1}} I(u_i^{t+1}|D_i^{t+1}) \prod_{k\in NLAF(i)} \Pr(xn^{k,t+1}|x_i^t,a_i,u_i^{t+1})$$

the probability of of the NLAFs.

In particular, we need to deal with the fact our definition of influence (5.3) can now be of the more complex form $I(u_i^{t+1}|D_i^{t+1},x_v^t,a_i^t,x_v^{t+1})$, as given by (5.5). This means that the NLAF probability

$\Pr(xn_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i,I_{\rightarrow i}^{t+1})$ given by (4.6) must be updated to deal with this new form, and this in turn implies that the definition of $\bar{T}_i(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t)$ per (4.7) needs to be updated too.

Let us start with the former. Like (3.8), this can now depend on ISDs from OLAFs xl_i^{t+1}

$$\Pr(xn_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,xl_i^{t+1},a_i,I_{\rightarrow i}^{t+1}) \triangleq \sum_{u_i^{t+1}=\langle y_u^t,a_u,y_u^{t+1}\rangle} I(u_i^{t+1}|D_i^{t+1},x_v^t,a_i^t,x_v^{t+1}) \Pr(xn_i^{t+1}|x_i^t,xl_i^{t+1},a_i,u_i^{t+1}), \quad (5.6)$$

with $\Pr(xn_i^{t+1}|x_i^t,xl_i^{t+1},a_i,u_i^{t+1})$ simply the product of CPTs of the NLAFs, as given by (3.8), but now restricted to only $y_i^t,y_i^{t+1},a_{\neq i}$ that are influence sources.

We are now in the position to define the IALM under intra-stage dependencies:

Definition 18 (IALM)**.** Given an LFM with intra-stage dependences, $\mathcal{M}^{LFM}$, and profile of policies for other agents $\pi_{\neq i}$, an *Influence-Augmented Local Model (IALM)* for agent i is a POMDP $\mathcal{M}_i^{IALM}(\mathcal{M}^{LFM},\pi_{\neq i}) = \left\langle \bar{\mathcal{S}},\mathcal{A}_i,\bar{T}_i,\bar{R}_i,\mathcal{O}_i,\bar{O}_i,h,b_i^{l,0} \right\rangle$, where

- $\bar{\mathcal{S}},\mathcal{A}_i,\bar{R}_i,\mathcal{O}_i,\bar{O}_i,h,b_i^{l,0}$ are identical to those in Definition 14,
- $\bar{T}_i$ is the transition function is defined as:

$$\begin{aligned}\bar{T}_i(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t) &\triangleq \Pr(x_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,a_i^t,I_{\rightarrow i}^{t+1})\mathbf{1}_{\{D_i^{t+2},d(x_i^t,a_i^t,x_i^{t+1},D_i^{t+1})\}} \\ &= \Pr(xn_i^{t+1}|\langle x_i^t,D_i^{t+1}\rangle,xl_i^{t+1},a_i,I_{\rightarrow i}^{t+1}) \Pr(xl_i^{t+1}|x_i^t,xn_i^{t+1},a_i)\mathbf{1}_{\{D_i^{t+2},d(x_i^t,a_i^t,x_i^{t+1},D_i^{t+1})\}}, \quad (5.7)\end{aligned}$$

with the first term is given by (5.6) and the second term is given by (3.7).

5.3 Planning in an IALM with ISDs

Since the only modifications that we needed to make to incorporate ISDs were in the transition function, the conclusions about how to plan in IALM made in Section 4.3 remain valid. In particular, the IALM is still a POMDP, with a well-defined belief-update function, and value functions. The solution of the IALM still gives the influence-based best-response value, defined in (4.13) as the value of the initial local-form belief: $V_i(I_{\rightarrow i}(\pi_{\neq i})) \triangleq V_i(b_i^{l,0})$.

6. Sufficiency of Influence-Based Abstraction

In this secion, we will show that influence-based abstraction is *completely lossless.* By that we mean that an IALM constructed according to Definition 18 can be used to accurately predict rewards and observations, and thus to compute an exact, optimal (best-response) value.

The latter is our main result, Theorem 1, which shows that the optimal values for the GFBRM and the IALM are equal, thus establishing that one can use the IALM to plan (or learn) in without any loss in value. In other words, it proves that the definition of influence actually constitutes a sufficient statistic for predicting the optimal value, and thus that resulting IALM actually achieves a best-response against the policy $\pi_{\neq i}$ that generated the influence $I_{\rightarrow i}(\pi_{\neq i})$.

Theorem 1. *For a finite-horizon POSG, the solution of the IALM for the incoming influence point $I_{\rightarrow i}(\pi_{\neq i})$ associated with any $\pi_{\neq i}$ achieves the same value $V_i(I_{\rightarrow i}(\pi_{\neq i}))$, given by* (4.13)*, as the best-response value $V_i(\pi_{\neq i})$, given by* (3.4)*, computed against $\pi_{\neq i}$ directly:*

$$\forall_{\pi_{\neq i}} \quad V_i(I_{\rightarrow i}(\pi_{\neq i})) = V_i(\pi_{\neq i}). \quad (6.1)$$

To prove this (in Section 6.4) we will show that the immediate reward terms and observation probabilities are equal (Section 6.3). In turn, to show this, we will need to show that transition probabilities are the same given a local-form belief and a global-form belief, which means that the local-form belief is a sufficient statistic to predict the next local state (Section 6.2). In order to allow the rewriting to take place, we first show how the global-form belief can be factorized.

We believe that this proof by itself is useful: it isolates the core technical property that needs to hold for sufficiency in Lemma 1 in Section 6.2. In this way it 1) conveys insight into the nature of how abstraction of state latent factors affects value, 2) provides a derivation that can be used to obtain simplifications of the definition of influence (Definition 17) in simpler cases, and 3) provides a recipe of how to prove similar results in problems which add even more complexities.

6.1 Factorization Of Global-Form Belief

In order to prove the equivalence of the GFBRM and the IALM, we will show that their value functions are the same. In order to do that, it will be necessary to decompose the global-form belief b_i^g in components.

To do that, we make use of the insight that, for any D_i^{t+1}, the law of total probability allows us to write

$$b_i^g(s^t,\vec{\theta}^t_{\neq i}) = \sum_{D_i^{t+1}} b_i(\langle x_i^t,y_i^t\rangle,\vec{\theta}^t_{\neq i},D_i^{t+1}) = \sum_{D_i^{t+1}} b_i(x_i^t,D_i^{t+1}) b_i(y_i^t,\vec{\theta}^t_{\neq i}|x_i^t,D_i^{t+1}). \tag{6.2}$$

Also, it is important to remember that the belief is *defined* as

$$b_i^g(s^t,\vec{\theta}^t_{\neq i}) \triangleq \Pr(s^t,\vec{\theta}^t_{\neq i}|\vec{\theta}^t_i,b^0,\pi_{\neq i}),$$

which means that in (6.2), the definitions of the components are

$$b_i(x_i^t,D_i^{t+1}) \triangleq \Pr(x_i^t,D_i^{t+1}|\vec{\theta}^t_i,b^0,\pi_{\neq i}), \tag{6.3}$$

$$b_i(y_i^t,\vec{\theta}^t_{\neq i}|x_i^t,D_i^{t+1}) \triangleq \Pr(y_i^t,\vec{\theta}^t_{\neq i}|x_i^t,D_i^{t+1},\vec{\theta}^t_i,b^0,\pi_{\neq i}). \tag{6.4}$$

These equations further clarify how to think about inclusion of actions a_i and observations o_i inside the d-separating set D_i^{t+1}: the belief *per definition* conditions on the history of actions and observations, as such these can be included in D_i^{t+1} without further problems. In particular, suppose that a_i^k is part of d-separating set D_i^{t+1}, then this will lead to $\Pr(x_i^t, \langle \dots a_i^k \dots \rangle \,|\, \langle \dots a_i^k \dots \rangle ,b^0,\pi_{\neq i})$ in (6.4). However, the interpretation is simply that this does not influence the probabilities, since $P(x|x) = 1$. Similarly, it would lead to a term $\Pr(y_i^t,\vec{\theta}^t_{\neq i}|x_i^t, \langle \dots a_i^k \dots \rangle, \langle \dots a_i^k \dots \rangle ,b^0,\pi_{\neq i})$ in (6.4). Again, this poses no problem, since $\Pr(y|x,x) = \Pr(y|x)$. However, let us repeat that we do need all observation relevant state factors in the local state: otherwise we cannot define the local observation model $\bar{O}_i$ and track the local-form belief $b_i(x_i^t,D_i^{t+1})$ (cf. Definition 9 and Definition 14).

6.2 Sufficiency for Prediction Local State Transitions

In this section, we show that the influence together with the local-form belief is sufficient to predict local state transitions. We first prove the following lemma, that shows that pairwise marginal distributions over states are the same in the IALM and the GFBRM. This will then be used in other proofs.

Lemma 1. *The joint distribution over current local state and next local state induced by a local-form belief is identical to that of the global-form belief:*

$$\forall_{\vec{\theta}_i^t} \forall_{x_i^t,x_i^{t+1}} \qquad \Pr(x_i^t,x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}) = \Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I^{t+1}_{\to i}),$$

where b_i^l, b_i^g *denote the for the local-form and global-form beliefs induced by* $\vec{\theta}_i^t$.

Proof. We assume arbitrary $\vec{\theta}_i^t, x_i^t, x_i^{t+1}$, and start with the left-hand side, which is given by (3.11):

$$\sum_{y_i^t} \sum_{a_{\neq i}} \Pr(x_i^{t+1} | s^t, a_i, a_{\neq i}) \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i} | \vec{\theta}_{\neq i}^t, \pi_{\neq i}) b_i^g(s^t, \vec{\theta}_{\neq i}^t)$$

$= \{\text{via } (6.2)\}$

$$\sum_{y_i^t} \sum_{a_{\neq i}} \Pr(x_i^{t+1} | s^t, a_i, a_{\neq i}) \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i} | \vec{\theta}_{\neq i}^t, \pi_{\neq i}) \left[\sum_{D_i^{t+1}} b_i(x_i^t, D_i^{t+1}) b_i(y_i^t, \vec{\theta}_{\neq i}^t | x_i^t, D_i^{t+1}) \right] \tag{6.5}$$

$= \{\text{via } (3.5)\}$

$$\sum_{y_i^t} \sum_{a_{\neq i}} \left[\sum_{y_i^{t+1}} \Pr(y_i^{t+1}, x_i^{t+1} | s^t, a_i, a_{\neq i}) \right] \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i} | \vec{\theta}_{\neq i}^t, \pi_{\neq i}) \sum_{D_i^{t+1}} b_i(x_i^t, D_i^{t+1}) b_i(y_i^t, \vec{\theta}_{\neq i}^t | x_i^t, D_i^{t+1}) \tag{6.6}$$

$= \{\text{via } (3.6)\}$

$$\sum_{y_i^t} \sum_{a_{\neq i}} \sum_{y_i^{t+1}} \Pr(xl_i^{t+1} | x_i^t, a_i, xn_i^{t+1}) \Pr(xn_i^{t+1} | x_i^t, xl_i^{t+1}, a_i, y_u^t, y_u^{t+1}, a_u) \Pr(y_i^{t+1} | x_i^t, y_i^t, a_i, a_{\neq i}, x_i^{t+1})$$
$$\sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i} | \vec{\theta}_{\neq i}^t, \pi_{\neq i}) \sum_{D_i^{t+1}} b_i(x_i^t, D_i^{t+1}) b_i(y_i^t, \vec{\theta}_{\neq i}^t | x_i^t, D_i^{t+1}) \tag{6.7}$$

$= \{\text{reordering terms}\}$

$$\sum_{D_i^{t+1}} \Pr(xl_i^{t+1} | x_i^t, a_i, xn_i^{t+1}) b_i(x_i^t, D_i^{t+1})$$
$$\left[\sum_{a_{\neq i}} \sum_{\vec{\theta}_{\neq i}^t} \sum_{y_i^t} \sum_{y_i^{t+1}} \Pr(xn_i^{t+1} | x_i^t, xl_i^{t+1}, a_i, y_u^t, y_u^{t+1}, a_u) \Pr(y_i^{t+1} | x_i^t, y_i^t, a_i, a_{\neq i}, x_i^{t+1}) \Pr(a_{\neq i} | \vec{\theta}_{\neq i}^t, \pi_{\neq i}) b_i(y_i^t, \vec{\theta}_{\neq i}^t | x_i^t, D_i^{t+1}) \right] \tag{6.8}$$

This equation has grouped together all the probabilities that are affected by the non-local part of the problem in the bracketed part. The terms before do not depend on the external part at all. We will now further investigate the externally influence (bracketed) part:

$$\sum_{a_{\neq i}} \sum_{\vec{\theta}^t_{\neq i}} \sum_{y^t_i} \sum_{y^{t+1}_i} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,u^{t+1}_i) \Pr(y^{t+1}_i|x^t_i,y^t_i,a_i,a_{\neq i},x^{t+1}_i)\pi_{\neq i}(a_{\neq i}|\vec{\theta}^t_{\neq i})b_i(y^t_i,\vec{\theta}^t_{\neq i}|x^t_i,D^{t+1}_i)$$

$$= \sum_{a_{\neq i}} \sum_{y^t_i} \sum_{y^{t+1}_i} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,u^{t+1}_i) \Pr(y^{t+1}_i|x^t_i,y^t_i,a_i,a_{\neq i},x^{t+1}_i) \sum_{\vec{\theta}^t_{\neq i}} \pi_{\neq i}(a_{\neq i}|\vec{\theta}^t_{\neq i})b_i(y^t_i,\vec{\theta}^t_{\neq i}|x^t_i,D^{t+1}_i)$$

=\{restricting to the intra-stage sources y^{t+1}_u and their intra-stage ancestors y^{t+1}_v, other factor's probabilities just sum to 1\}

$$\sum_{a_{\neq i}} \sum_{y^t_i} \sum_{y^{t+1}_u} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,u^{t+1}_i) \sum_{y^{t+1}_v} \Pr(y^{t+1}_u,y^{t+1}_v|x^t_i,y^t_i,a_i,a_{\neq i},x^{t+1}_i) \sum_{\vec{\theta}^t_{\neq i}} \pi_{\neq i}(a_{\neq i}|\vec{\theta}^t_{\neq i})b_i(y^t_i,\vec{\theta}^t_{\neq i}|x^t_i,D^{t+1}_i)$$

=\{restricting to a_v,x^{t+1}_v that actually influence y^{t+1}_u. I.e., v denotes other 'second order' sources\}

$$\sum_{a_{\neq i}} \sum_{y^t_i} \sum_{y^{t+1}_u} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,u^{t+1}_i) \sum_{y^{t+1}_v} \Pr(y^{t+1}_u,y^{t+1}_v|x^t_v,y^t_v,a_i,a_v,x^{t+1}_v) \sum_{\vec{\theta}^t_{\neq i}} \pi_{\neq i}(a_{\neq i}|\vec{\theta}^t_{\neq i})b_i(y^t_i,\vec{\theta}^t_{\neq i}|x^t_i,D^{t+1}_i)$$

\{cf notes about $\Pr(y^{t+1}_u,y^{t+1}_v|x^t_v,y^t_v,a_i,a_v,x^{t+1}_v)$ \} (6.9)

=\{pushing in summations, recall $u^{t+1}_i = \left\langle y^t_u,a_u,y^{t+1}_u \right\rangle$\}

$$\sum_{u^{t+1}_i} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,u^{t+1}_i) \sum_{a_v} \sum_{y^t_v} \sum_{y^{t+1}_v} \Pr(y^{t+1}_u,y^{t+1}_v|x^t_v,y^t_v,a_i,a_v,x^{t+1}_v) \sum_{\vec{\theta}^t_u} \sum_{\vec{\theta}^t_v} \pi_{u\cup v}(a_u,a_v|\vec{\theta}^t_u,\vec{\theta}^t_v)b_i(y^t_u,y^t_v,\vec{\theta}^t_u,\vec{\theta}^t_v|x^t_i,D^{t+1}_i) \tag{6.10}$$

=\{let $w = u \cup v$\}

$$\sum_{u^{t+1}_i} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,u^{t+1}_i) \sum_{a_v} \sum_{y^t_v} \sum_{y^{t+1}_v} \Pr(y^{t+1}_u,y^{t+1}_v|x^t_v,y^t_v,a_i,a_v,x^{t+1}_v) \sum_{\vec{\theta}^t_w} \pi_w(a_w|\vec{\theta}^t_w,\pi_w)b_i(y^t_w,\vec{\theta}^t_w|x^t_i,D^{t+1}_i)$$

=\{since $b_i(y^t_w,\vec{\theta}^t_w|x^t_i,D^{t+1}_i) \triangleq \Pr(y^t_w,\vec{\theta}^t_w|x^t_i,D^{t+1}_i,\vec{\theta}^t_i,b^0,\pi_{\neq i}) \overset{\{\text{def. of d-set (5.2)}\}}{=} \Pr(y^t_w,\vec{\theta}^t_w|D^{t+1}_i,b^0,\pi_{\neq i})$ \}

$$\sum_{u^{t+1}_i} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,u^{t+1}_i) \sum_{\left\langle y^t_v,a_v,y^{t+1}_v \right\rangle} \Pr(y^{t+1}_u,y^{t+1}_v|x^t_v,y^t_v,a_i,a_v,x^{t+1}_v) \sum_{\vec{\theta}^t_w} \pi_w(a_w|\vec{\theta}^t_w) \Pr(y^t_w,\vec{\theta}^t_w|D^{t+1}_i,b^0,\pi_{\neq i}). \tag{6.11}$$

We can now apply the definition of influence (Definition 17) to (6.11), which yields

$$= \sum_{u^{t+1}_i = \left\langle y^t_u,a_u,y^{t+1}_u \right\rangle} \Pr(xn^{t+1}_i|x^t_i,xl^{t+1}_i,a_i,y^t_u,a_u,y^{t+1}_u) I(u^{t+1}_i|D^{t+1}_i,x^t_v,a_i,x^{t+1}_v), \tag{6.12}$$

which is the definition (5.6) of $\Pr(xn^{t+1}_i|\langle x^t_i,D^{t+1}_i\rangle,xl^{t+1}_i,a_i,I^{t+1}_{\to i})$.

Substituting (6.12) back in (6.8) we get

$$\sum_{D^{t+1}_i} \Pr(xl^{t+1}_i|x^t_i,x^{t+1}_i,xn^{t+1}_i,a_i)b_i(x^t_i,D^{t+1}_i) \left[\Pr(xn^{t+1}_i|\langle x^t_i,D^{t+1}_i\rangle,xl^{t+1}_i,a_i,I^{t+1}_{\to i})\right]$$

$=$\{via 5.7 \}

$$\sum_{D^{t+1}_i} \Pr(x^{t+1}_i|x^t_i,D^{t+1}_i,a_i,I^{t+1}_{\to i})b_i(x^t_i,D^{t+1}_i) \overset{\{\text{via (4.12)}\}}{=} \Pr(x^t_i,x^{t+1}_i|b^l_i,a^t_i,I^{t+1}_{\to i}),$$

which concludes the proof. □

Lemma 2. *A local-form belief is a sufficient statistic for prediction the next local state. That is, when b^l_i, b^g_i denote the for the local-form and global-form beliefs induced by the same action-observation history $\vec{\theta}^t_i$, we have that:*

$$\forall_{\vec{\theta}^t_i} \forall_{x^{t+1}_i} \qquad \Pr(x^{t+1}_i|b^g_i,a_i,\pi_{\neq i}) = \Pr(x^{t+1}_i|b^l_i,a_i,I^{t+1}_{\to i}). \tag{6.13}$$

Proof. This follows directly from Lemma (1):

$$\Pr(x_i^{t+1}|b_i^l,a_i^t,I_{\to i}^{t+1}) = \sum_{x_i^t} \Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I_{\to i}^{t+1}) = \sum_{x_i^t} \Pr(x_i^t,x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}) = \Pr(x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}).$$

□

6.3 Sufficiency for Predicting Rewards and Observations

Given that we established that local-form beliefs in an IALM are sufficient to predict local-state transitions, we can now also establish their sufficiency for predicting rewards and observations.

Lemma 3. *The local-form belief is a sufficient statistic to predict the immediate reward. That is*

$$\forall_{\vec{\theta}_i} \forall_{a_i} \qquad R_i(b_i^g,a_i) = R_i(b_i^l,a_i) \tag{6.14}$$

where b_i^l, b_i^g denote the for the local-form and global-form beliefs induced by $\vec{\theta}_i$.

Proof. Comparing equations (3.10) and (4.11), we see that this holds if $\Pr(x_i^t,x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}) = \Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I_{\to i}^{t+1})$. This is precisely what Lemma 1 shows. □

Lemma 4. *The local-form belief is a sufficient statistic for predicting the observation. That is:*

$$\forall_{\vec{\theta}_i} \forall_{a_i,o_i} \qquad \Pr(o_i|b_i^g,a_i) = \Pr(o_i|b_i^l,a_i), \tag{6.15}$$

where b_i^l, b_i^g denote the for the local-form and global-form beliefs induced by $\vec{\theta}_i$.

Proof. Comparing equations (3.12) and (4.9), we see that equality holds if $\Pr(x_i^{t+1}|b_i^g,a_i,\pi_{\neq i}) = \Pr(x_i^{t+1}|b_i^l,a_i,I_{\to i}^{t+1})$; this is exactly what Lemma 2 shows. □

6.4 Proof of Theorem 1: Sufficiency for Predicting Optimal (Best-Response) Value

The values in (6.1) are defined as the value of the initial beliefs, cf. equations (4.13) and (3.4), and we will show that these values are equal. The proof is by induction over the horizon, where the base case is given by the last stage.

Base Case. Assume an arbitrary last-stage AOH, $\vec{\theta}_i^{h-1}$, and let b_i^l, b_i^g denote the for the local-form and global-form beliefs induced by it. Their respective values are given by

$$V_i(b_i^g) = \max_{a_i} R_i(b_i^g,a_i),$$

$$V_i(b_i^l) = \max_{a_i} R_i(b_i^l,a_i).$$

So we need to show that the predicted immediate rewards are equal. This, however, is exactly what Lemma 3 shows.

Induction Step. The induction hypothesis is that, for stage $t+1$,

$$\forall_{\vec{\theta}_i^{t+1}} \qquad V_i^{t+1}(b_i^{l,t+1}) = V_i^{t+1}(b_i^{g,t+1}),$$

where we write $b_i^{l,t+1}, b_i^{g,t+1}$ are the local-form and global-form beliefs induced by $\vec{\theta}_i^{t+1}$.

Now we need to prove that $V_i^t(b_i^l) = V_i^t(b_i^g)$, for all $\vec{\theta}_i^t$. We will show this by proving that equality hold for the Q-values. Assume and arbitrary $\vec{\theta}_i^t$, its Q-values are given by (3.1):

$$Q_i(b_i^g,a_i) = R_i(b_i^g,a_i) + \gamma \sum_{o_i} \Pr(o_i|b_i^g,a_i) V_i^{t+1}(BU(b_i^g,a_i,o_i))$$

By the induction hypothesis, we get

$$Q_i^t(b_i^g,a_i) = R_i(b_i^g,a_i) + \sum_{o_i} \Pr(o_i|b_i^g,a_i) V_i^{t+1}(BU(b_i^l,a_i,o_i)).$$

(Note that $BU(b_i^g,a_i,o_i)$ and $BU(b_i^l,a_i,o_i)$ are the local-form and global-form beliefs induced by the same next-stage history $\vec{\theta}_i^{t+1} = (\vec{\theta}_i^{t},a_i,o_i)$, and hence the induction hypothesis applies.)

So, in order to show that this is equal to

$$Q_i^t(b_i^l,a_i) = R_i(b_i^l,a_i) + \sum_{o_i} \Pr(o_i|b_i^l,a_i) V_i^{t+1}(BU(b_i^l,a_i,o_i))$$

we need to show equality for both the immediate rewards, $R_i(b_i^g,a_i) = R_i(b_i^l,a_i)$, and the observation probabilities, $\Pr(o_i|b_i^g,a_i) = \Pr(o_i|b_i^l,a_i)$. The former was shown in Lemma 3 and the latter was shown in Lemma 4. □

7. Conclusion, Discussion and Future Work

This paper makes a theoretical contribution to the field of decision making in factored multiagent settings. It defines a formulation of 'influence' that enables an agent to perform, under certain assumptions, a lossless abstraction of the decision making problem it faces. That is, we prove that, for a given abstraction in terms of a *local-form* model, an *influence point* is a sufficient statistic for the part of the problem that is abstracted away. The local-form model and influence point together induce what we call an *influence-augmented local model*: a local model that is sufficient to compute a best response.

The proof of sufficiency is not only a validation of the theory. It also serves a practical purpose: it isolates the core technical property that needs to hold for sufficiency in Lemma 1 in Section 6.2. In this way it 1) conveys insight into the nature of how abstraction of state latent factors affects value, 2) provides a derivation that can be used to obtain simplifications of the definition of influence (Definition 17) in simpler cases, and 3) provides a recipe of how to prove similar results in problems which add even more complexities.

We emphasize that this definition of influence is not a magic bullet: while the influence-augmented local model is sufficient to compute a best-response locally, the computation of the required influence point itself is an intractable inference problem in general. However, in certain specific cases where this problem *is* feasible it can enable faster best-response computations and search for multiagent plans via *influence search* [45, 47]. As such, an important direction of future work would investigate how the definition of influence presented in this paper can support influence search in more general settings.

Moreover, even in cases where influences are intractable to compute, the concept forms the basis for principled approximations. For instance, by being *optimistic* with respect to the influence sources, one is able to compute upper bounds on the optimal value of Dec-POMDPs with hundreds of agents, thus leading to firm guarantees on the quality of heuristic solutions [31]. Furthermore, there is preliminary evidence, in the context of deep reinforcement learning, that such approximate versions of influence may in some problems improve learning, both in terms of speed as performance [10]. As such, a fruitful direction of research is to better understand such approximate characterization of influence. This document has provided the foundations for such an exploration.

Finally, we note that even though the discussion in this paper was based on the more general case of multiagent systems, there is nothing that stops us from applying IBA in complex systems with just a single agent. If this can lead to benefits remains to be demonstrated.

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Appendix A. Proofs & Derivations

Here we give proofs and derivations of a number of results. These are referred from the main text, and will be stated here without further explanation.

A.1 GFBRMs

A.1.1 Expected Reward

$$
\begin{aligned}
R_i(b_i^g,a_i^t) &= \mathbf{E}_{\bar{s}_i^t\sim b_i^g,\bar{s}_i^{t+1}\sim\bar{T}(\bar{s}_i^t,a_i^t,\cdot)}\left[\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1})\right] \\
&= \sum_{\bar{s}_i^t} b_i^g(\bar{s}_i^t) \sum_{\bar{s}_i^{t+1}} \bar{T}(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t)\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1}) \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle) \sum_{\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1}\rangle} \Pr(\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1}\rangle|\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i)\bar{R}_i(\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i,\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1}\rangle) \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle) \sum_{s^{t+1}}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}} \Pr(s^{t+1},a_{\neq i},o_{\neq i}^{t+1}|\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i)R_i(s^t,a_i,a_{\neq i},s^{t+1}) \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle) \sum_{s^{t+1}}\sum_{a_{\neq i}} \Pr(s^{t+1},a_{\neq i}|\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i)R_i(s^t,a_i,a_{\neq i},s^{t+1}) \\
&= \sum_{s^t}\sum_{s^{t+1}}\sum_{a_{\neq i}} \Pr(s^{t+1}|s^t,a)R_i(s^t,a,s^{t+1}) \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t)b_i^g(s^t,\vec{\theta}_{\neq i}^t)
\end{aligned}
$$

A.1.2 Expected Observation Probability

$$
\begin{aligned}
\Pr(o_i^{t+1}|b_i^g,a_i^t) &= \mathbf{E}_{\bar{s}_i^t\sim b_i^g,\bar{s}_i^{t+1}\sim\bar{T}(\bar{s}_i^t,a_i^t,\cdot)}\left[\bar{O}(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1})\right] \\
&= \sum_{\bar{s}_i^t} b_i^g(\bar{s}_i^t)\sum_{\bar{s}_i^{t+1}}\bar{T}(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t)\bar{O}(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1}) \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle)\sum_{\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1}\rangle}\Pr(\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1}\rangle|\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i)\Pr(o_i|a_i,\langle s^{t+1},\vec{\theta}_{\neq i}^{t+1}\rangle) \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle)\sum_{s^{t+1}}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}}\Pr(s^{t+1},a_{\neq i},o_{\neq i}^{t+1}|\langle s^t,\vec{\theta}_{\neq i}^t\rangle,a_i)\Pr(o_i|a_i,a_{\neq i},s^{t+1},o_{\neq i}^{t+1}) \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle)\sum_{s^{t+1}}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}}\Pr(s^{t+1}|s^t,a_{\neq i},a_i)\Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t,\pi_{\neq i})\Pr(o_{\neq i}^{t+1}|a_i,a_{\neq i},s^{t+1}) \\
&\quad \Pr(o_i|a_i,a_{\neq i},s^{t+1},o_{\neq i}^{t+1}) \qquad \text{(A.1)} \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle)\sum_{s^{t+1}}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}}\Pr(s^{t+1}|s^t,a_{\neq i},a_i)\Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t,\pi_{\neq i})\Pr(o_{\neq i}^{t+1}|a_i,a_{\neq i},s^{t+1}) \\
&\quad \frac{\Pr(o_i,o_{\neq i}^{t+1}|a_i,a_{\neq i},s^{t+1})}{\Pr(o_{\neq i}^{t+1}|a_i,a_{\neq i},s^{t+1})} \\
&= \sum_{\langle s^t,\vec{\theta}_{\neq i}^t\rangle} b_i^g(\langle s^t,\vec{\theta}_{\neq i}^t\rangle)\sum_{s^{t+1}}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}}\Pr(s^{t+1}|s^t,a)\Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t,\pi_{\neq i})\Pr(o_i,o_{\neq i}^{t+1}|a_i,a_{\neq i},s^{t+1}) \\
&= \sum_{s^t}\sum_{s^{t+1}}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}}\Pr(s^{t+1}|s^t,a)\Pr(o^{t+1}|a,s^{t+1})\sum_{\vec{\theta}_{\neq i}^t}\Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t,\pi_{\neq i})b_i^g(s^t,\vec{\theta}_{\neq i}^t) \qquad \text{(A.2)}
\end{aligned}
$$

A.2 LFMs

A.2.1 Expected Reward

Starting with (3.2):

$$
\begin{aligned}
R_i(b_i^g,a_i) &= \sum_{s^t}\sum_{s^{t+1}}\sum_{a_{\neq i}} \Pr(s^{t+1}|s^t,a) R_i(s^t,a,s^{t+1}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{s^t}\sum_{s^{t+1}}\sum_{a_{\neq i}} \Pr(s^{t+1}|s^t,a_i,a_{\neq i}) R_i(s^t,a,s^{t+1}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \{\text{restrict to actual dependencies of } R_i\} \\
&\quad \sum_{s^t}\sum_{s^{t+1}}\sum_{a_{\neq i}} \Pr(s^{t+1}|s^t,a_i,a_{\neq i}) R_i(x_i^t,a_i,x_i^{t+1}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{s^t}\sum_{a_{\neq i}}\sum_{x_i^{t+1},y_i^{t+1}} \Pr(x_i^{t+1},y_i^{t+1}|s^t,a_i,a_{\neq i}) R_i(x_i^t,a_i,x_i^{t+1}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \{\text{via (3.5)}\} \\
&\quad \sum_{x_i^t,y_i^t}\sum_{a_{\neq i}}\sum_{x_i^{t+1}} \Pr(x_i^{t+1}|s^t,a_i,a_{\neq i}) R_i(x_i^t,a_i,x_i^{t+1}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{x_i^t}\sum_{x_i^{t+1}} R_i(x_i^t,a_i,x_i^{t+1}) \sum_{y_i^t}\sum_{a_{\neq i}} \Pr(x_i^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{x_i^t}\sum_{x_i^{t+1}} R_i(x_i^t,a_i,x_i^{t+1}) \Pr(x_i^t,x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}), \qquad \text{(A.3)}
\end{aligned}
$$

where we implicitly defined (remember $s^t = \langle x_i^t, y_i^t \rangle$)

$$
\Pr(x_i^t,x_i^{t+1}|b_i^g,a_i^t,\pi_{\neq i}) \triangleq \sum_{y_i^t}\sum_{a_{\neq i}} \Pr(x_i^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \qquad \text{(A.4)}
$$

A.2.2 Expected Observation Probability

$$
\begin{aligned}
\Pr(o_i^{t+1}|b_i^g,a_i) &= \sum_{s^t}\sum_{s^{t+1}}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}} \Pr(s^{t+1}|s^t,a) \Pr(o|a,s^{t+1}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{s^{t+1}}\sum_{s^t}\sum_{a_{\neq i}}\sum_{o_{\neq i}^{t+1}} \Pr(o_i^{t+1},o_{\neq i}^{t+1}|a_i,a_{\neq i},s^{t+1}) \Pr(s^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \{\text{marginalize}\} \\
&\quad \sum_{s^{t+1}}\sum_{s^t}\sum_{a_{\neq i}} \Pr(o_i^{t+1}|a_i,a_{\neq i},s^{t+1}) \Pr(s^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \{\text{restrict to actual dependencies}\} \\
&\quad \sum_{s^{t+1}}\sum_{s^t}\sum_{a_{\neq i}} \Pr(o_i^{t+1}|a_i,x_i^{t+1}) \Pr(s^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{x_i^{t+1},y_i^{t+1}}\sum_{s^t}\sum_{a_{\neq i}} \Pr(o_i^{t+1}|a_i,x_i^{t+1}) \Pr(x_i^{t+1},y_i^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{x_i^{t+1}} \Pr(o_i^{t+1}|a_i,x_i^{t+1}) \sum_{s^t}\sum_{y_i^{t+1}}\sum_{a_{\neq i}} \Pr(x_i^{t+1},y_i^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}^t_{\neq i}} \Pr(a_{\neq i}|\vec{\theta}^t_{\neq i},\pi_{\neq i}) b_i^g(s^t,\vec{\theta}^t_{\neq i}) \\
&= \sum_{x_i^{t+1}} \Pr(o_i^{t+1}|a_i,x_i^{t+1}) \Pr(x_i^{t+1}|b_i^g,a_i,\pi_{\neq i}) \qquad \text{(A.5)}
\end{aligned}
$$

where we implicitly defined

$$\Pr(x_i^{t+1}|b_i^g,a_i) \triangleq \sum_{s^t}\sum_{a_{\neq i}} \Pr(x_i^{t+1}|s^t,a_i,a_{\neq i}) \sum_{\vec{\theta}_{\neq i}^t} \Pr(a_{\neq i}|\vec{\theta}_{\neq i}^t \pi_{\neq i}) b_i^g(s^t,\vec{\theta}_{\neq i}^t). \tag{A.6}$$

A.3 IALMs

A.3.1 Expected Reward

$$\begin{aligned}
R_i(b_i^l,a_i^t) &= \mathbf{E}_{\bar{s}_i^t\sim b_i^l,\bar{s}_i^{t+1}\sim\bar{T}(\bar{s}_i^t,a_i^t,\cdot)}\left[\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1})\right] \\
&= \sum_{\bar{s}_i^t} b_i^l(\bar{s}_i^t) \sum_{\bar{s}_i^{t+1}} \bar{T}(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i^t)\bar{R}_i(\bar{s}_i^t,a_i^t,\bar{s}_i^{t+1}) \\
&= \sum_{x_i^t,D_i^{t+1}} b_i^l(x_i^t,D_i^{t+1}) \sum_{x_i^{t+1},D_i^{t+2}} \Pr(x_i^{t+1},D_i^{t+2}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1})R_i(x_i^t,a_i^t,x_i^{t+1}) \\
&= \sum_{x_i^t,D_i^{t+1}} b_i^l(x_i^t,D_i^{t+1}) \sum_{x_i^{t+1}} \Pr(x_i^{t+1}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1})R_i(x_i^t,a_i^t,x_i^{t+1}) \\
&= \sum_{x_i^t}\sum_{x_i^{t+1}} R_i(x_i^t,a_i^t,x_i^{t+1}) \left[\sum_{D_i^{t+1}} \Pr(x_i^{t+1}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1})b_i^l(x_i^t,D_i^{t+1})\right] \\
&= \sum_{x_i^t}\sum_{x_i^{t+1}} R_i(x_i^t,a_i^t,x_i^{t+1}) \Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I_{\to i}^{t+1})
\end{aligned} \tag{A.7}$$

where we implicitly defined

$$\Pr(x_i^t,x_i^{t+1}|b_i^l,a_i^t,I_{\to i}^{t+1}) \triangleq \sum_{D_i^{t+1}} \Pr(x_i^{t+1}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1})b_i^l(x_i^t,D_i^{t+1}) \tag{A.8}$$

(consistent with 4.12).

A.3.2 Expected Observation Probability

$$\begin{aligned}
\Pr(o_i^{t+1}|b_i^l,a_i^t) &= \mathbf{E}_{\bar{s}_i^t\sim b_i^l,\bar{s}_i^{t+1}\sim\bar{T}(\bar{s}_i^t,a_i,\cdot)}\left[\bar{O}(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1})\right] \\
&= \sum_{\bar{s}_i^t} b_i^l(\bar{s}_i^t) \sum_{\bar{s}_i^{t+1}} \bar{T}(\bar{s}_i^{t+1}|\bar{s}_i^t,a_i)\bar{O}(o_i^{t+1}|a_i^t,\bar{s}_i^{t+1}) \\
&= \sum_{x_i^t,D_i^{t+1}} b_i^l(x_i^t,D_i^{t+1}) \sum_{x_i^{t+1},D_i^{t+2}} \Pr(x_i^{t+1},D_i^{t+2}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1}) \Pr(o_i^{t+1}|a_i^t,x_i^{t+1}) \\
&= \sum_{x_i^t,D_i^{t+1}} b_i^l(x_i^t,D_i^{t+1}) \sum_{x_i^{t+1}} \Pr(x_i^{t+1}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1}) \Pr(o_i^{t+1}|a_i^t,x_i^{t+1}) \\
&= \sum_{x_i^{t+1}} \Pr(o_i^{t+1}|a_i^t,x_i^{t+1}) \left[\sum_{x_i^t,D_i^{t+1}} \Pr(x_i^{t+1}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1})b_i^l(x_i^t,D_i^{t+1})\right] \\
&= \sum_{x_i^{t+1}} \Pr(o_i^{t+1}|a_i^t,x_i^{t+1}) \Pr(x_i^{t+1}|b_i^l,a_i^t,I_{\to i}^{t+1}),
\end{aligned} \tag{A.9}$$

where we implicitly defined

$$\Pr(x_i^{t+1}|b_i^l,a_i,I_{\to i}^{t+1}) \triangleq \sum_{x_i^t,D_i^{t+1}} \Pr(x_i^{t+1}|x_i^t,D_i^{t+1},a_i^t,I_{\to i}^{t+1}) b_i^l(x_i^t,D_i^{t+1}). \tag{A.10}$$

(consistent with 4.10).

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